

ARITHMETIC SIEGEL WEIL FORMULA ON $X_0(N)$

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ABSTRACT. In this paper, we proved an arithmetic Siegel-Weil formula and the modularity of some arithmetic theta function on the modular curve $X_0(N)$ when N is square free. In the process, we also construct some generalized Delta function for $\Gamma_0(N)$ and proved some explicit Kronecker limit formula for Eisenstein series on $X_0(N)$.

1. INTRODUCTION

It is well-known that there is a deep connection between the leading term of some analytic functions and the arithmetics, such as the class number formula, Birch and Swinnerton-Dyer conjecture, Block-Kato conjecture and the Siegel-Weil formula. Little is known or understood is about the possible connection between the second term of these functions and arithmetic although it started to change in this century. The most famous one is the Kronecker limit formula:

$$E(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \Im(\gamma z)^s = 1 + \frac{1}{6}(\log |\Delta(\tau) \Im(z)^6|)s + O(s^2).$$

We refer to [Si] for its proof and its beautiful application to class numbers. In 2004, Kulda, Rapoport and Yang ([KRY1]) discovered another second term identity of some Eisenstein series of weight $3/2$ —the so-called arithmetic Siegel-Weil formula. Roughly speaking, they defined an arithmetic function—a generating function $\hat{\phi}_{KRY}(\tau)$ of a family of arithmetic divisors in a Shimura curve. They proved that its degree is the special value of some Eisenstein series $\mathcal{E}(\tau, s)$ (weight $3/2$) at $s = 1$ and that arithmetic intersection with the (normalized) metrized Hodge bundle on the Shimura curve is the derivative of the same Eisenstein series $\mathcal{E}(\tau, s)$ at $s = 1$ (second term). This case is different from the Kronecker formula in two ways. First, the leading term is already deeply connected with arithmetics by the Siegel-Weil formula. Second,

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the second term (derivative) is found to be deeply related with the Gille-Soule height pairing on a Shimura curve. Its analogue in $X_0(4)$ was worked out later by Kudla and Yang, and was reported in [Ya1]. In this case, the Eisenstein series is Zagier's famous Eisenstein series [HZ] of weight $3/2$. In [BF2], Bruinier and Funke gave a different proof of the main result of [Ya1] using theta lifting. Colmez conjecture [Col] can also be viewed as an second term of 'CM' Hecke L -functions $L'(0, \chi)$ in terms of Faltings' height. We should mention the breakthrough formula of Zhiwei Yun and Wei Zhang which relates the n -th central derivative of the L -function of an automorphic representation on GL_2 over a function field and height pairing of some cycles in middle dimension on some Drinfeld space [YZ]. We also mention the beautiful second term identity in the Siegel-Weil formula (see for example [GQT] and references there), although it has different flavor.

Later in the book [KRY2, Chapter 4], Kudla proved that the arithmetic theta function $\widehat{\phi}_{KRY}$ is modular. In this paper, we will prove both the arithmetic Siegel-Weil formula and the modularity of a similar arithmetic theta function in the case of modular curve $X_0(N)$. The complication comes mainly from the cusps, and we need to understand the behavior of Kudla's Green functions at cusps carefully. The metrized Hodge bundle has log singularity at cusps presents another complication. The method in [KRY1] does not seem to extend to this case easily. Instead, we will use theta lifting method following [BF2]. After the arithmetic Siegel-Weil formula is proved, the modularity theorem follows the same method of [KRY2, Chapter 4] with a little modification. In the process of proving the arithmetic Siegel-Weil formula, we also obtained some explicit Kronecker limit formula for Eisenstein series of weight 0 for $\Gamma_0(N)$, which should be of independent interest. In particular, the explicit modular form (denoted by Δ_N), as rational section of Hodge bundle, plays an essential role in proving the arithmetic Siegel-Weil formula. Now we set up notation and describe the main results in a little more detail.

Let

$$(1.1) \quad L = \left\{ w = \begin{pmatrix} b & \frac{-a}{N} \\ c & -b \end{pmatrix} \in M_2(\mathbb{Z}) \mid a, b, c \in \mathbb{Z} \right\}$$

with quadratic form $Q(w) = N \det w = -Nb^2 + ac$, and let L^\sharp be its dual lattice. Then $\mathrm{Spin}(V) \cong \mathrm{SL}_2$ acts on V by conjugation, and the associated Hermitian symmetric domain \mathbb{D} is isomorphic to the upper half plane \mathbb{H} . Since $\Gamma_0(N)$ preserves L and acts on L^\sharp/L trivially, we can and will identify $X_0(N)$ with the compactification of the open orthogonal Shimura curve $\Gamma_0(N) \backslash \mathbb{D}$ (see Section 2 for detail).

For each $\mu \in L^\sharp/L$, denote $L_\mu = \mu + L$, and

$$L_\mu[n] = \{w \in L_\mu : Q(w) = n\}.$$

For $\mu \in L^\sharp/L$ and a positive rational number $n \in Q(\mu) + \mathbb{Z}$, we define divisors

$$(1.2) \quad Z(n, \mu) := \sum_{w \in \Gamma_0(N) \backslash L_\mu[n]} Z(w).$$

When $\mu = \mu_r = \text{diag}(\frac{r}{2N}, -\frac{r}{2N})$, this divisor is the same as the Heegner divisors $P_{D,r} + P_{D,-r} \in \text{CH}^1(X_0(N))$ in [GKZ], where $D = -4Nn$ is a discriminant. For a positive real number $v > 0$, let $\Xi(n, \mu, v)$ be the well-known Kudla Green function for $Z(n, \mu)$ in the open modular curve $Y_0(N)$ as defined in [Ku1] (see (5.4) for precise definition). The behavior of $\Xi(n, \mu, v)$ at cusps is complicated and not studied before. In Sections 5 and 6, we will prove that it is smooth and of exponential decay when $D = -4Nn$ is not a square, and has singularity along the the cusps (Section 6) when $D \neq 0$ is a square. Even worse, when $D = 0$ (which forces $\mu = 0$), $\Xi(0, 0, v)$ has log-log singularity in the sense of [BKK] (see Section 4).

Let $\mathcal{X}_0(N)$ be the canonical integral model over \mathbb{Z} of $X_0(N)$ as defined in [KM] (see Section 6). For a point $x \in \mathcal{X}_0(N)$ over a field, since $\{\pm 1\} \subseteq \text{Aut}(x)$ always, we count x with multiplicity $\frac{2}{|\text{Aut}(x)|}$ for convenience. Let $\mathcal{Z}(n, \mu)$ be the Zariski closure of $Z(n, \mu)$ in $\mathcal{X}_0(N)$, and we obtain a family of arithmetic divisors $\widehat{\mathcal{Z}}(n, \mu, v)$ in $\widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}_0(N))$ —arithmetic Chow group with real coefficients in the sense of Gillet-Soule as follows for $n \neq 0$:

$$\widehat{\mathcal{Z}}(n, \mu, v) = \begin{cases} (\mathcal{Z}(n, \mu), \Xi(n, \mu, v)) & \text{if } n > 0, \\ (0, \Xi(n, \mu, v)) & \text{if } n < 0, D \neq \square, \\ (g(n, \mu, v) \sum_{P \text{ cusps}} \mathcal{P}, \Xi(n, \mu, v)) & \text{if } n < 0, D = \square. \end{cases}$$

Here $g(n, \mu, v)$ is some real number defined in Theorem 6.3, and \mathcal{P} is the Zariski closure of the cusp P in $\mathcal{X}_0(N)$. Let $\widehat{\omega}_N$ be the metrized Hodge bundle on $\mathcal{X}_0(N)$ with the normalized Pettersson metric. It has log singularity at cusps in the sense of Kühn (see Section 4). Its associated arithmetic divisor has log-log singularity at the cusps. It turns out magically that the modified arithmetic divisor

$$(1.3) \quad \widehat{\mathcal{Z}}(0, 0, v) = \widehat{\mathcal{Z}}(0, 0, v)^{\text{Naive}} - 2\widehat{\omega}_N - \sum_{p|N} \mathcal{X}_p^0 - (0, \log(\frac{v}{N}))$$

belongs to $\widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}_0(N))$ (Proposition 6.6). Here \mathcal{X}_p^0 (resp. \mathcal{X}_p^∞) is the irreducible component of $\mathcal{X}_0(N) \pmod{p}$ containing the reduction of

the cusp P_0 (resp. P_∞). One of the main purposes of this paper is to prove the following analogue of the modularity theorem in [KRY2, Chapter 4].

Theorem 1.1. *The arithmetic theta generating function (for $\tau = u + iv$, and $q_\tau = e(\tau) = e^{2\pi i\tau}$)*

$$(1.4) \quad \widehat{\phi}(\tau) = \sum_{\mu \in L^\sharp/L} \sum_{n \in Q(\mu) + \mathbb{Z}} \widehat{\mathcal{Z}}(n, \mu, v) q_\tau^n e_\mu,$$

is a vector valued modular form for Γ' of weight $\frac{3}{2}$, valued in $\mathbb{C}[L^\sharp/L] \otimes \widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}_0(N))$. Here Γ' is the metaplectic cover of $\text{SL}_2(\mathbb{Z})$ which acts on $\mathbb{C}[L^\sharp/L]$ via the Weil representation ρ_L (see (2.2)) and acts on the arithmetic Chow group trivially. Finally $\{e_\mu : \mu \in L^\sharp/L\}$ is the standard basis of $\mathbb{C}[L^\sharp/L]$.

As in [KRY2, Chapter 4], we will need the decomposition theorem of the arithmetic Chow group $\widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}_0(N))$ and a couple of arithmetic intersection modularity theorems which are of independent interest. Actually, we can identify some of the resulting modular forms, which we now briefly describe.

Let

$$E_L(\tau, s) = \sum_{\gamma' \in \Gamma'_\infty \backslash \Gamma'} (v^{\frac{s-1}{2}} e_{\mu_0}) |_{3/2} \gamma'$$

be a vector valued Eisenstein series of weight $3/2$, where the Petersson slash operator is defined on functions $f : \mathbb{H} \rightarrow \mathbb{C}[L^\sharp/L]$ by

$$(f |_{3/2} \gamma')(\tau) = \phi(\tau)^{-3} \rho_L^{-1}(\gamma') f(\gamma\tau),$$

where $\gamma' = (\gamma, \phi) \in \Gamma'$. Let

$$(1.5) \quad \mathcal{E}_L(\tau, s) = -\frac{s}{4} \pi^{-s-1} \Gamma(s) \zeta^{(N)}(2s) N^{\frac{1}{2} + \frac{3}{2}s} E_L(\tau, s)$$

be its normalization, where

$$\zeta^{(N)}(s) = \zeta(s) \prod_{p|N} (1 - p^{-s}).$$

Remark 1.2. In the work [KRY1] and [Ya1], the critic point of Eisenstein series is $s = \frac{1}{2}$. In our paper, for the convenience of computation, we define $E_L(\tau, s)$ by a shift of s .

Similar to [KRY2, Chapter 4], the main ingredients in proving the modularity theorem is the following arithmetic intersection theorem, which is analogue of the main result in [KRY1] and generalization of [Ya1] and [BF2, Section 7]. The third formula is usually called an

arithmetic Siegel-Weil formula while the first one (degree formula) is a geometric Siegel-Weil formula.

Theorem 1.3. *Let the notation be as above. Then*

$$\begin{aligned}\langle \widehat{\phi}(\tau), a(1) \rangle_{GS} &= \frac{1}{\varphi(N)} \mathcal{E}_L(\tau, 1), \\ \langle \widehat{\phi}(\tau), \mathcal{X}_p^0 \rangle_{GS} &= \langle \widehat{\phi}(\tau), \mathcal{X}_p^\infty \rangle_{GS} = \frac{1}{\varphi(N)} \mathcal{E}_L(\tau, 1) \log p, \quad p|N\end{aligned}$$

and

$$\langle \widehat{\phi}(\tau), \widehat{\omega}_N \rangle_{GS} = -\frac{1}{\varphi(N)} \left(\mathcal{E}'_L(\tau, 1) + \sum_{p|N} \frac{p}{p-1} \mathcal{E}_L(\tau, 1) \log p \right).$$

Here $a(1) = (0, 1) \in \widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}_0(N))$.

There are three main ingredients in the proof of Theorem 1.3. The first is to analyze and understand the behavior of Kudla's green function $\Xi(n, \mu, v)$ for all pairs $(n, \mu) \in \mathbb{Q} \times L^\sharp/L$ with $Q(\mu) \equiv n \pmod{1}$, in particular when $D = -4Nn \geq 0$ is a square. Here $v > 0$ is a constant. This occupies full Section 5 (general case) and the first part of Section 6. The upshot is an honest definition of the arithmetic divisors $\widehat{\mathcal{Z}}(n, \mu, v)^{\text{Naive}}$ in Theorem 6.3, its modification $\widehat{\mathcal{Z}}(n, \mu, v)$ in (6.10), and the generating function $\widehat{\phi}(\tau)$ above. This part is perhaps the most technical part.

To understand $\widehat{\omega}_N$, we actually construct an explicit rational section of ω_N^k , which is isomorphic to the line bundle of modular forms of weight k for $k = 12\varphi(N)$ (the Euler φ -function), i.e., an explicit modular form Δ_N of weight k for $\Gamma_0(N)$ as follows:

$$(1.6) \quad \Delta_N(z) = \prod_{t|N} \Delta(tz)^{a(t)}$$

with

$$a(t) = \sum_{r|t} \mu\left(\frac{t}{r}\right) \mu\left(\frac{N}{r}\right) \frac{\varphi(N)}{\varphi\left(\frac{N}{r}\right)},$$

where $\mu(n)$ is the Möbius function. This is inspired by Kühn's early work on self-intersection of $\widehat{\omega}_N$ with $N = 1$ using the well-known Delta function Δ . One complication here is that Δ_N has vertical components, see Lemma 6.4. This means that we will need to deal with self-intersections of vertical components (see Section 7).

These ingredients are enough for the first two identities of Theorem 1.3. To prove the last identity, we further need to compute the infinity part of the arithmetic intersection, which boils down essentially

to self-intersection of $\widehat{\omega}_N$, intersection of vertical components, and the following integral, which can be viewed as a theta lifting:

$$(1.7) \quad I(\tau, \log \|\Delta_N\|) = \int_{X_0(N)} \log \|\Delta_N\| \Theta_L(\tau, z).$$

Here $\Theta_L(\tau, z)$ is the two variable geometric theta kernel of Kudla and Millson defined by (2.6), and the Petersson norm is renormalized as

$$(1.8) \quad \|f(z)\| = |f(z)(4\pi e^{-C}y)^{\frac{k}{2}}| = e^{-\frac{kC}{2}} \|f(z)\|_{Pet},$$

with $C = \frac{\log 4\pi + \gamma}{2}$. The theta function $\Theta_L(\tau, z)$ is a vector valued modular form for τ of weight $3/2$ and modular function for the variable z valued in $\Omega^{1,1}(X_0(N))$ for $\Gamma_0(N)$.

To connect this integral with $\mathcal{E}'_L(\tau, 1)$, we follow Bruinier and Funke's idea in [BF2] in two steps, given by the following two theorems, which are of independent interest.

Theorem 1.4. (*Theta lifting of Eisenstein series*) *Let*

$$(1.9) \quad E(N, z, s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} (\Im(\gamma z))^s,$$

be the Eisenstein series of weight 0 for $\Gamma_0(N)$, and let

$$(1.10) \quad \mathcal{E}(N, z, s) := N^{2s} \pi^{-s} \Gamma(s) \zeta^{(N)}(2s) E(N, z, s)$$

be its normalization. Then

$$I(\tau, \mathcal{E}(N, z, s)) = I(\tau, \mathcal{E}(N, w_N z, s)) = \zeta^*(s) \mathcal{E}_L(\tau, s),$$

where $w_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ and $\zeta^(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$.*

Theorem 1.5. (*Kronecker Limit formula for $\Gamma_0(N)$*) *Let the notation be as above. Then one has*

$$\lim_{s \rightarrow 1} \left(\mathcal{E}(N, z, s) - \varphi(N) \zeta^*(2s-1) \right) = -\frac{1}{12} \log \left(y^{6\varphi(N)} \mid \Delta_N(z) \mid \right),$$

and

$$\lim_{s \rightarrow 1} \left(\mathcal{E}(N, w_N z, s) - \varphi(N) \zeta^*(2s-1) \right) = -\frac{1}{12} \log \left(y^{6\varphi(N)} \mid \Delta_N^0(z) \mid \right),$$

where $\Delta_N^0 = \Delta_N | w_N$.

Combining the previous theorems, we obtain

Theorem 1.6. *One has (for $k = 12\varphi(N)$)*

$$I(\tau, 1) = \frac{2}{\varphi(N)} \mathcal{E}_L(\tau, 1)$$

and

$$I(\tau, \log \|\Delta_N\|) = I(\tau, \log \|\Delta_N^0\|) = -12\mathcal{E}'_L(\tau, 1).$$

This paper is organized in two parts as follows. In Part I, we prove Theorem 1.4 after setting up notation and introduce the theta lifting (2.9) in Section 2. In Section 3, we study some basic properties of Δ_N and prove the Kronecker limit formula Theorem 1.5 and then Theorem 1.6. We also prove some properties of Δ_N needed in Part II.

In Part II, we first review arithmetic divisors with log-log singularity, metrized line bundles with log singularity, and arithmetic intersection in Section 4 following Kühn [Kü2] and Burgos Gil, Kramer and U. Kühn [BKK]. In Section 5, we study the behavior of Kudla Green functions at cusps in a more general setting (see Theorem 5.1). In Section 6, we focus on the modular curve $\mathcal{X}_0(N)$ for square free N , and prove Theorem 6.3. We also decompose Theorem 1.3 into two propositions, which we prove in Section 6, and a ‘horizontal intersection’ theorem Theorem 6.9, which we will prove in Section 7. In last section, we will prove the modularity theorem (Theorem 1.1).

Finally, we remark that the technical condition N being square free is only needed in the arithmetic part, mainly to avoid the complication of special fiber of $\mathcal{X}_0(N)$ at $p^2|N$ when N is not square free. A different way to prove the modularity theorem is to first prove a modularity theorem for a similar generating function with Bruinier Green functions and then to show the generating function of the difference of the two Green functions is modular as done in recent work of Ehlen and Sankaran [ES].

Acknowledgments. Add later.

Part 1. Theta lifting and Kronecker limit formula

2. BASIC SET-UP AND THETA LIFTING

Let

$$(2.1) \quad V = \left\{ w = \begin{pmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{pmatrix} \in M_2(\mathbb{Q}) : \text{tr}(w) = 0 \right\},$$

with the quadratic form $Q(w) = N \det w = -Nw_2w_3 - Nw_1^2$, which has signature $(1, 2)$. Let L be the even integral lattice defined in the introduction with dual lattice L^\sharp . We will identify

$$\mathbb{Z}/2N\mathbb{Z} \cong L^\sharp/L, \quad r \mapsto \mu_r = \begin{pmatrix} \frac{r}{2N} & 0 \\ 0 & -\frac{r}{2N} \end{pmatrix}.$$

Let $G = \text{SL}_2 \cong \text{Spin}(V)$ act on V by conjugation, i.e., $g.w = gw g^{-1}$. Notice that $\Gamma_0(N)$ preserves L and acts on L^\sharp/L trivially. Let \mathbb{D} be the

Hermitian domain of positive real lines in $V_{\mathbb{R}}$:

$$\mathbb{D} = \{z \in V_{\mathbb{R}}; \dim z = 1 \text{ and } (\ , \)|_z > 0\}.$$

The following lemma can be easily checked and is left to the reader.

Lemma 2.1. *For $z = x + iy$, define*

$$w(z) = \frac{1}{\sqrt{N}y} \begin{pmatrix} -x & z\bar{z} \\ -1 & x \end{pmatrix}.$$

Then $z \mapsto [w(z)] = \mathbb{R}w(z)$ gives an $G(\mathbb{R})$ -invariant isomorphism between the upper half plane \mathbb{H} and \mathbb{D} . It induces thus an isomorphism between $Y_0(N) = \Gamma_0(N) \backslash \mathbb{H}$ and $\Gamma_0(N) \backslash \mathbb{D}$.

Let $X_0(N)$ be the usual compactification of $Y_0(N)$. Let $\text{Mp}_{2,\mathbb{R}}$ be the metaplectic double cover of $\text{SL}_2(\mathbb{R})$, which can be realized as pairs $(g, \phi(g, \tau))$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, $\phi(g, \tau)$ is a holomorphic function of $\tau \in \mathbb{H}$ such that $\phi(g, \tau)^2 = j(g, \tau) = c\tau + d$. Let Γ' be the preimage of $\Gamma = \text{SL}_2(\mathbb{Z})$ in $\text{Mp}_{2,\mathbb{R}}$, then Γ' is generated by

$$S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right) \quad T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right).$$

We denote the standard basis of $S_L = \mathbb{C}[L^\sharp/L]$ by $\{e_\mu = L_\mu = \mu + L : \mu \in L^\sharp/L\}$. Then there is a Weil representation ρ_L of Γ' on $\mathbb{C}[L^\sharp/L]$ such that ([Bo1])

$$(2.2) \quad \begin{aligned} \rho_L(T)e_\mu &= e(Q(\mu))e_\mu, \\ \rho_L(S)e_\mu &= \frac{e(\frac{1}{8})}{\sqrt{|L^\sharp/L|}} \sum_{\mu' \in L^\sharp/L} e(-(\mu, \mu'))e_{\mu'}. \end{aligned}$$

This Weil representation ρ_L is naturally connected to the Weil representation ω of $\text{Mp}_{2,\mathbb{A}}$ on $S(V_{\mathbb{A}})$, see [BHY] for explanation.

Following Kudla and Millson ([KM], [BF2, section 3]), we decompose for $z = x + iy \in \mathbb{H}$,

$$V_{\mathbb{R}} = \mathbb{R}w(z) \oplus w(z)^\perp, \quad w = w_z + w_{z^\perp},$$

and define $R(w, z) = -(w_{z^\perp}, w_{z^\perp})$, and the majorant

$$(w, w)_z = (w_z, w_z) + R(w, z).$$

Since $Q(w(z)) = 1$, it is easy to check

$$(2.3) \quad \begin{aligned} R(w, z) &= \frac{1}{2}(w, w(z))^2 - (w, w), \\ (w, w)_z &= (w, w(z))^2 - (w, w). \end{aligned}$$

For $w = \begin{pmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{pmatrix} \in V_{\mathbb{R}}$, we have

$$(2.4) \quad (w, w(z)) = -\frac{\sqrt{N}}{y}(w_3 z \bar{z} - w_1(z + \bar{z}) - w_2).$$

Let $\mu(z) = \frac{dx dy}{y^2}$,

$$(2.5) \quad \begin{aligned} \varphi^0(w, z) &= \left((w, w(z))^2 - \frac{1}{2\pi} \right) e^{-2\pi R(w, z)} \mu(z), \\ \varphi(w, \tau, z) &= e(Q(w)\tau) \varphi^0(\sqrt{v}w, z), \end{aligned}$$

which is a Schwartz function on $V_{\mathbb{R}}$ valued in $\Omega^{1,1}(\mathbb{D})$ constructed by Kudla and Millson in [KM]. Finally, let

$$(2.6) \quad \Theta_L(\tau, z) = \sum_{\mu \in L^{\sharp}/L} \theta_{\mu}(\tau, z) e_{\mu}$$

be the vector valued Kudla-Millson theta function, which is nonholomorphic modular form of weight $3/2$ of (Γ', ρ_L) with respect to the variable τ with values in $\Omega^{1,1}(X_{\Gamma})$. It is $\Gamma_0(N)$ -invariant as a function of z . Here

(2.7)

$$\begin{aligned} \theta_{\mu}(\tau, z) &= \sum_{n, \mu} \sum_{w \in L_{\mu}[n]} \varphi(w, \tau, z) \\ &= \sum_{n \in \mathbb{Q}_{\geq 0}, Q(\mu) \equiv n \pmod{1}} \omega(n, \mu, v)(z) q^n + \begin{cases} 0 & \text{if } \mu \neq 0, \\ -\frac{1}{2\pi} \mu(z) & \text{if } \mu = 0, \end{cases} \end{aligned}$$

with $(q = q_{\tau} = e(\tau))$

$$(2.8) \quad \omega(n, \mu, v)(z) = \sum_{0 \neq w \in L_{\mu}[n]} \varphi^0(v^{\frac{1}{2}}w, z) \in \Omega^{1,1}(X_{\Gamma}).$$

The following result of Funke about behavior of θ_{μ} as z goes to the boundary (cusp) is important to our definition of theta lifting.

Proposition 2.2. [BF2, Proposition 4.1] *Fix $\mu \in L^{\sharp}/L$ and $\tau \in \mathbb{H}$. Let $l = \sigma_l(\infty)$ be a cusp of $X_0(N)$. As a function of $z = x + iy \in \mathbb{H} = \mathbb{D}$, the theta function*

$$\theta_{\mu}(\tau, \sigma_l z) = O(e^{-Cy^2}), \text{ as } y \longrightarrow \infty$$

holds uniformly in x for some constant $C > 0$.

For a (non-holomorphic) modular function $f(z)$ for $\Gamma_0(N)$ (viewed as subgroup of the Spin group) with moderate growth, the theta lifting

$$(2.9) \quad I(\tau, f) = \int_{\Gamma_0(N) \backslash \mathbb{D}} f(z) \Theta_L(\tau, z) = \sum_{\mu \in L^\sharp / L} I_\mu(\tau, f) e_\mu$$

is absolutely convergent by Proposition 2.2 and is a (non-holomorphic) weight $3/2$ modular form of Γ' with values in $\mathbb{C}[L^\sharp / L]$.

Proof of Theorem 1.4: First, we compute the theta series:

$$\begin{aligned} \theta_{\mu_r}(\tau, z) &= \sum_{w \in L_{\mu_r}} \varphi(w, \tau, z) \\ &= \sum_{w_1 \in \mathbb{Z} + \frac{r}{2N}, n, w_3 \in \mathbb{Z}} \left(\frac{v}{Ny^2} (N(w_3 z \bar{z} - w_1(z + \bar{z})) - n)^2 - \frac{1}{2\pi} \right) \\ &\quad \times e(-N\bar{\tau}w_1^2) e(-\bar{\tau}w_3 n) e\left(\frac{iv}{2Ny^2} (N(w_3 z \bar{z} - w_1(z + \bar{z})) - n)^2 \right) \mu(z). \end{aligned}$$

Let

$$f(X) = \left(\frac{vX^2}{Ny^2} - \frac{1}{2\pi} \right) e(-\bar{\tau}w_3 X) e\left(\frac{ivX^2}{2Ny^2} \right),$$

then

$$\widehat{f}(m) = \int_{-\infty}^{\infty} f(X) e(-mX) dX = -\frac{N^{\frac{3}{2}} y^3}{v^{\frac{3}{2}}} (\bar{\tau}w_3 + m)^2 e\left(\frac{iNy^2}{2v} (\bar{\tau}w_3 + m)^2 \right).$$

Denoting $t = N(w_3 z \bar{z} - w_1(z + \bar{z}))$, and applying the Poisson summation formula, we obtain

$$\begin{aligned} \theta_{\mu_r}(\tau, z) &= \sum_{w_1 \in \mathbb{Z} + \frac{r}{2N}, m, w_3 \in \mathbb{Z}} e(-N\bar{\tau}w_1^2) e(-\bar{\tau}w_3 t) \widehat{f}(m) e(-mt) \mu(z) \\ &= -\frac{N^{\frac{3}{2}} y^3}{v^{\frac{3}{2}}} \sum_{w_1 \in \mathbb{Z} + \frac{r}{2N}, m, w_3 \in \mathbb{Z}} (\bar{\tau}w_3 + m)^2 e(-N\bar{\tau}(w_1 - w_3 x)^2) \\ &\quad \times e(2N(w_1 - w_3 m/2)mx) \exp\left(-\frac{\pi Ny^2}{v} |m + w_3 \tau|^2 \right) \mu(z). \end{aligned}$$

As in [Bo1, Section 4], we define for $\alpha, \beta \in \mathbb{Q}$

$$(2.10) \quad \Theta_L(\tau, \alpha, \beta) = \sum_{r \in \mathbb{Z}/2N} \sum_{w_1 \in \frac{r}{2N} + \mathbb{Z}} e(-\bar{\tau}(w_1 + \beta)^2) e(-\alpha(2w_1 + \beta)) e_{\mu_r}.$$

For $\gamma' = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right) \in \Gamma'$, it is easy to check

$$(2.11) \quad \Theta_L(\tau, ndx, -ncx) = (c\bar{\tau} + d)^{-\frac{1}{2}} \rho_L^{-1}(\gamma') \Theta_L(\gamma' \tau, nx, 0).$$

We continue the calculation:

$$\begin{aligned}
& \Theta_L(\tau, z) \\
= & -\frac{N^{\frac{3}{2}}y^3}{v^{\frac{3}{2}}} \sum_{m, w_3 \in \mathbb{Z}} (\bar{\tau}w_3 + m)^2 e\left(-\frac{\pi Ny^2}{v}|m+w_3\tau|^2\right) \Theta_L(\tau, mx, -w_3x)\mu(z) \\
= & -\frac{N^{\frac{3}{2}}y^3}{v^{\frac{3}{2}}} \sum_{n=1}^{\infty} n^2 \sum_{c, d \in \mathbb{Z}, (c, d)=1} (c\bar{\tau} + d)^2 e\left(-\frac{\pi Ny^2 n^2}{v}|c\bar{\tau}+d|^2\right) \Theta_L(\tau, ndx, -ncx)\mu(z) \\
= & -\frac{N^{\frac{3}{2}}y^3}{v^{\frac{3}{2}}} \sum_{n=1}^{\infty} n^2 \sum_{\gamma' \in \Gamma'_\infty \setminus \Gamma'} (c\bar{\tau} + d)^{\frac{3}{2}} e\left(-\frac{\pi Ny^2 n^2}{v}|c\bar{\tau}+d|^2\right) \rho_L^{-1}(\gamma') \Theta_L(\gamma'\tau, nx, 0)\mu(z).
\end{aligned}$$

Unfolding the integral, we have for $\Re(s) > 1$

$$\begin{aligned}
I(\tau, E(N, z, s)) &= \int_{\Gamma_\infty \setminus \mathbb{H}} \Theta_L(\tau, z) y^s \\
= & -v^{-\frac{3}{2}} N^{\frac{3}{2}} \sum_{n=1}^{\infty} n^2 \sum_{\gamma' \in \Gamma'_\infty \setminus \Gamma'} (c\bar{\tau} + d)^{3/2} \int_0^\infty e\left(-\frac{\pi Ny^2 n^2}{v}|c\bar{\tau}+d|^2\right) y^{s+1} dy \\
& \times \rho_L^{-1}(\gamma') \int_0^1 \Theta_L(\gamma'\tau, nx, 0) dx.
\end{aligned}$$

It is easy to check that

$$\int_0^1 \Theta_L(\gamma'\tau, nx, 0) dx = e_{\mu_0}.$$

So

$$\begin{aligned}
& \int_{\Gamma_\infty \setminus \mathbb{H}} \Theta_L(\tau, z) y^s \\
= & -\frac{1}{2} v^{-\frac{3}{2}} N^{\frac{3}{2}} \sum_{n=1}^{\infty} n^2 \sum_{\gamma' \in \Gamma'_\infty \setminus \Gamma'} \frac{v^{\frac{s+2}{2}} (c\bar{\tau} + d)^{3/2} \Gamma\left(\frac{s}{2} + 1\right)}{\pi^{\frac{s+2}{2}} |c\tau + d|^{s+2} N^{\frac{s+2}{2}} n^{s+2}} \rho_L^{-1}(\gamma') e_{\mu_0} \\
= & -\frac{1}{2} N^{\frac{1-s}{2}} \zeta(s) \Gamma\left(\frac{s}{2} + 1\right) \sum_{\gamma' \in \Gamma'_\infty \setminus \Gamma'} \frac{v^{\frac{s-1}{2}} (c\bar{\tau} + d)^{3/2}}{\pi^{\frac{s}{2}+1} |c\tau + d|^{s+2}} \rho_L^{-1}(\gamma') e_{\mu_0} \\
= & -N^{\frac{1-s}{2}} \frac{s}{4\pi} \zeta^*(s) \sum_{\gamma' \in \Gamma'_\infty \setminus \Gamma'} \left(v^{\frac{s-1}{2}} e_{\mu_0}\right)_{|3/2, L} \gamma'.
\end{aligned}$$

In summary, we have proved

$$I(\tau, E(N, z, s)) = -N^{\frac{1-s}{2}} \frac{s}{4\pi} \zeta^*(s) E_L(\tau, s).$$

From normalization we obtain

$$(2.12) \quad I(\tau, \mathcal{E}(N, z, s)) = \zeta^*(s) \mathcal{E}_L(\tau, s).$$

It is easy to check by definition that

$$\theta_L(\tau, z) = \theta_L(\tau, w_N(z)).$$

This implies that

$$I(\tau, \mathcal{E}(N, w_N(z), s)) = I(\tau, \mathcal{E}(N, z, s)).$$

This proves the theorem.

Taking residue of both sides of the equation (2.12) at $s = 1$, we have the following result.

Corollary 2.3.

$$(2.13) \quad I(\tau, 1) = \frac{2}{\varphi(N)} \mathcal{E}_L(\tau, 1).$$

3. KRONECKER LIMIT FORMULA FOR THE GROUP $\Gamma_0(N)$

We need some preparation before proving the Kronecker Limit formula for $\Gamma_0(N)$. These auxiliary results will also be used in Section 6 and should be of independent interest.

Let

$$(3.1) \quad C_N(n) = \sum_{a=1, (a, N)=1}^N e\left(\frac{an}{N}\right)$$

be the Ramanujan sum. It has the following properties according to Kluver ([Kl, p.411]).

Lemma 3.1. (*Kluver*) *Let $t = (N, n)$ be the greatest common divisor of N and n . Then one has*

$$C_N(n) = \frac{\varphi(N)}{\varphi(\frac{N}{t})} C_{\frac{N}{t}}(1)$$

$$C_N(n) = \sum_{r|t} \mu\left(\frac{N}{r}\right) r.$$

Here φ is the classical Euler φ -function, and $\mu(t)$ is the well-known Möbius function. In particular, one has $C_N(1) = \mu(N)$.

Lemma 3.2. *For a positive integer N and a divisor t of N , let*

$$a_N(t) = \sum_{r|t} \mu\left(\frac{t}{r}\right) \mu\left(\frac{N}{r}\right) \frac{\varphi(N)}{\varphi(\frac{N}{r})}$$

be as in the introduction. Then the following are true.

- (1) If $Q \parallel N$, i.e., $Q \mid N$ and $(Q, N/Q) = 1$, write $t = t_1 t_2$. Then $a_N(t) = a_Q(t_1) a_{N/Q}(t_2)$.
- (2) One has

$$\begin{aligned} \sum_{t \mid N} a_N(t) &= \varphi(N), \\ \sum_{t \mid N} t a_N(t) &= N \varphi(N) \prod_{p \mid N} (1 + p^{-1}), \\ \sum_{t \mid N} t^{-1} a_N(t) &= 0 \quad \text{when } N > 1. \end{aligned}$$

Proof. (1) is clear. For (2), we check the second identity and leave the others to the reader. We drop the subscript N from now on as N will be fixed. One has

$$\begin{aligned} \sum_{t \mid N} t a(t) &= \sum_{t \mid N} t \sum_{r \mid t} \mu\left(\frac{t}{r}\right) \mu\left(\frac{N}{r}\right) \frac{\varphi(N)}{\varphi\left(\frac{N}{r}\right)} \\ &= \varphi(N) \sum_{r \mid N} \frac{\mu\left(\frac{N}{r}\right)}{\varphi\left(\frac{N}{r}\right)} \sum_{t \mid \frac{N}{r}} r t \mu(t) \quad (\text{replacing } t \text{ by } r t) \\ &= N \varphi(N) \sum_{\substack{r \mid N \\ r \text{ square free}}} \frac{\mu(r)}{r \varphi(r)} \sum_{t \mid r} t \mu(t) \quad (\text{replacing } N/r \text{ by } r) \\ &= N \varphi(N) \sum_{\substack{r \mid N \\ r \text{ square free}}} \frac{1}{r} = N \varphi(N) \prod_{p \mid N} (1 + p^{-1}). \end{aligned}$$

□

Proposition 3.3. Let $\Delta_N(z)$ be defined as in (1.6). Then

$$\Delta_N(z) = q_z^{N \varphi(N) \prod_{p \mid N} (1 + p^{-1})} \prod_{n \geq 1} (1 - q_z^n)^{24 C_N(n)}.$$

Proof. Let

$$\tilde{\Delta}_N(z) = \prod_n (1 - q_z^n)^{C_N(n)}, \text{ and } \tilde{\Delta}(z) = \prod_{n=1}^{\infty} (1 - q_z^n).$$

Suppose that there are numbers $b(t)$ with

$$\tilde{\Delta}_N(z) = \prod_{t \mid N} \tilde{\Delta}(t z)^{b(t)},$$

which implies by Lemma 3.1

$$\begin{aligned}
\prod_{t|N} \prod_{(n, \frac{N}{t})=1} (1 - q_z^{tn})^{\frac{\varphi(N)}{\varphi(N/t)} \mu(N/t)} &= \prod_{t|N} \prod_n (1 - q_z^{tn})^{b(t)} \\
&= \prod_{t|N} \prod_{t'| \frac{N}{t}} \prod_{(n, \frac{N}{tt'})=1} (1 - q_z^{tt'n})^{b(t)} \\
&= \prod_{r|N} \prod_{t|r} \prod_{(n, \frac{N}{r})=1} (1 - q_z^{rn})^{b(t)} \\
&= \prod_{r|N} \prod_{(n, \frac{N}{r})=1} (1 - q_z^{rn})^{\sum_{t|r} b(t)}.
\end{aligned}$$

So for every $r|N$, one has

$$(3.2) \quad \sum_{t|r} b(t) = \frac{\varphi(N)}{\varphi(N/r)} \mu(N/r).$$

By Möbius inverse formula, one has

$$b(t) = \sum_{r|t} \mu\left(\frac{t}{r}\right) \mu\left(\frac{N}{r}\right) \frac{\varphi(N)}{\varphi\left(\frac{N}{r}\right)} = a(t).$$

So we have proved that

$$\tilde{\Delta}_N(z) = \prod_{t|N} \tilde{\Delta}(tz)^{a(t)}.$$

Combining this with Lemma 3.2 (2), we obtain the lemma. \square

Recall ([Mi]) that cusps of $X_0(N)$ are given by $P_{\frac{aQ}{N}} = \frac{aQ}{N}$, where $Q|N$ and $a \in (\mathbb{Z}/(Q, N/Q)\mathbb{Z})^\times$. In particular, when $Q||N$, i.e., $Q|N$ and $(Q, N/Q) = 1$, there is a unique cusp $P_{\frac{Q}{N}}$ associated to it. $Q = 1$ is associated to $P_\infty = P_{\frac{1}{N}}$, and $Q = N$ is associated to $P_0 = P_1$. Assume $Q||N$, and let

$$W_Q = \begin{pmatrix} \alpha & \beta \\ \gamma \frac{N}{Q} & Q\delta \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma \frac{N}{Q} & Q\delta \end{pmatrix} \in \Gamma_0(N/Q)$$

be an Atkin-Lehner involution matrix with $w_Q(P_\infty) = P_{\frac{Q}{N}}$. Notice that when N is a square free, $P_{\frac{Q}{N}}$, $Q|N$, give all the cusps of $X_0(N)$. The following proposition gives Fourier expansion of Δ_N at cusps associated to $Q||N$.

Proposition 3.4. *Assume $Q \parallel N$. For $t \mid N$, write $t_0 = (t, Q)$ for their greatest common divisor. Then*

$$\Delta_N | W_Q(z) = C_Q \prod_{t \mid N} \Delta\left(\frac{t}{t_0} \frac{Q}{t_0} z\right)^{a_N(t)}$$

where

$$C_Q = Q^{6\varphi(N)} \prod_{t_0 \mid Q} t_0^{-12\varphi(\frac{N}{Q})a_Q(t_0)}.$$

In particular, $\text{ord}_p C_Q = 0$ for $p \nmid Q$. Moreover, $\Delta_N(z)$ does not vanish at cusps associated to $Q \parallel N$ (with $Q \neq 1$).

Proof. Write $k = 12\varphi(N)$, $t = t_0 t_1$, and $Q = t_0 Q_1$. Then

$$\begin{aligned} \Delta_N | W_Q(z) &= \frac{Q^{\frac{k}{2}}}{(\gamma N z + Q \delta)^k} \prod_{t \mid N} \Delta\left(\frac{\alpha Q t z + t \beta}{\gamma N z + Q \delta}\right)^{a_N(t)} \\ &= \frac{Q^{\frac{k}{2}}}{(\gamma N z + Q \delta)^k} \prod_{t \mid N} \Delta\left(\frac{\alpha t_0(t_1 Q_1 z) + t_1 \beta}{\gamma \frac{N}{t_1 Q}(t_1 Q_1 z) + Q_1 \delta}\right)^{a_N(t)} \\ &= A_Q \prod_{t \mid N} \Delta(t_1 Q_1 z)^{a_N(t)}, \end{aligned}$$

where (recall Lemma 3.2)

$$A_Q = Q^{\frac{k}{2}} \prod_{t \mid Q} t_0^{-12a_N(t)} = C_Q.$$

On the other hand, the leading q -power exponent of $\Delta_N | W_Q$ is given by the above calculation (recall again Lemma 3.2)

$$\begin{aligned} \sum_{t \mid Q} t_1 Q_1 a_N(t) &= \sum_{t_0 \mid Q} \frac{Q}{t_0} a_Q(t_0) \sum_{t_1 \mid \frac{N}{Q}} t_1 a_{\frac{N}{Q}}(t_1) \\ &= \begin{cases} 0 & \text{if } Q > 1, \\ N\varphi(N) \prod_{p \mid N} (1 + p^{-1}) & \text{if } Q = 1. \end{cases} \end{aligned}$$

This proves the result. \square

Proof of Theorem 1.5: Recall the Whittaker function ([WD, Chapter 2]) for $y > 0$ and $\alpha, \beta \in \mathbb{C}$:

$$(3.3) \quad W(y, \alpha, \beta) = \Gamma(\beta)^{-1} \int_0^\infty (1+h)^{\alpha-1} h^{\beta-1} e^{-yh} dh.$$

Define

$$t_n(y, \alpha, \beta) = \begin{cases} i^{\beta-\alpha}(2\pi)^{\alpha+\beta} n^{\alpha+\beta-1} e^{-2\pi n y} \Gamma(\alpha)^{-1} W(4\pi n y, \alpha, \beta), & \text{if } n > 0, \\ i^{\beta-\alpha}(2\pi)^{\alpha+\beta} |n|^{\alpha+\beta-1} e^{-2\pi |n| y} \Gamma(\beta)^{-1} W(4\pi |n| y, \beta, \alpha), & \text{if } n < 0, \\ i^{\beta-\alpha}(2\pi)^{\alpha+\beta} \Gamma(\alpha)^{-1} \Gamma(\beta)^{-1} \Gamma(\alpha + \beta - 1) (4\pi y)^{1-\alpha-\beta}, & \text{if } n = 0. \end{cases}$$

One has by calculation ($z = x + iy \in \mathbb{H}$)

$$\begin{aligned} E(N, z, s) &= \frac{y^s}{2\zeta^{(N)}(2s)} \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (N,n)=1}} \frac{1}{|mNz + n|^{2s}} \\ &= y^s + \frac{y^s}{N^{2s} \zeta^{(N)}(2s)} \sum_{m=1}^{\infty} \sum_{\substack{1 \leq a < N \\ (a,N)=1}} \sum_{j \in \mathbb{Z}} |mz + \frac{a}{N} + j|^{-2s} \\ &= y^s + \frac{y^s}{N^{2s} \zeta^{(N)}(2s)} \sum_{n \in \mathbb{Z}} \sum_{m=1}^{\infty} t_n(my, s, s) \sum_{a \in (\mathbb{Z}/N)^\times} e\left(\frac{n(mNx + a)}{N}\right). \end{aligned}$$

Write

$$\mathcal{E}(N, z, s) = N^{2s} \pi^{-s} \Gamma(s) \zeta^{(N)}(2s) E(N, z, s) = \sum_{k \in \mathbb{Z}} a_k(z, s) e(kx).$$

Then one has first that

$$a_0(z, s) = N^{2s} \pi^{-s} \Gamma(s) \zeta^{(N)}(2s) y^s + \varphi(N) y^s \pi^{-s} \frac{(2\pi)^{2s} \Gamma(2s-1) (4\pi y)^{1-2s} \zeta(2s-1)}{\Gamma(s)}.$$

Simple calculation gives

$$(3.4) \quad a_0(z, s) = \varphi(N) \left(\frac{1}{2(s-1)} - \frac{\log y}{2} - \frac{\log 4\pi - \gamma}{2} + \frac{\pi}{6} y N \prod_{p|N} (1+p^{-1}) \right) + O(s-1).$$

On the other hand,

$$(3.5) \quad \zeta^*(2s-1) = \frac{1}{2(s-1)} - \frac{1}{2} (\log 4\pi - \gamma) + O(s-1).$$

So

$$(3.6) \quad \lim_{s \rightarrow 1} (a_0(z, s) - \varphi(N) \zeta^*(2s-1)) = \varphi(N) \left(-\frac{\log y}{2} + \frac{\pi}{6} y N \prod_{p|N} (1+p^{-1}) \right).$$

For $k > 0$, one has

$$\begin{aligned} a_k(z, s) &= y^s \pi^{-s} \Gamma(s) \sum_{mn=k} t_n(my, s, s) \sum_{a=1, (a, N)=1}^N e(nmx + anN^{-1}) \\ &= y^s \pi^{-s} \Gamma(s) (2\pi)^{2s} \frac{W(4\pi ky, s, s)}{\Gamma(s) e^{2\pi ky}} \sum_{mn=k} n^{2s-1} C_N(n). \end{aligned}$$

As

$$W(4k\pi y, 1, 1) = \frac{1}{4k\pi y},$$

one has

$$(3.7) \quad a_k(z, 1) = \frac{e^{-2\pi ky}}{k} \sum_{n|k} n C_N(n).$$

From the definition of $a_k(z, s)$ and $t_n(y, \alpha, \beta)$ as above, we know that $a_{-k}(z, 1) = a_k(z, 1)$. Therefore,

$$\begin{aligned} \mathcal{E}(N, z, s) &= \sum_{k=-\infty}^{\infty} a_k(z, s) e(kx) \\ &= a_0(z, s) + \sum_{k>0} \frac{1}{k} \sum_{n|k} n C_N(n) q_z^k + \sum_{k>0} \frac{1}{k} \sum_{n|k} n C_N(n) \bar{q}_z^k + O(s-1) \\ &= a_0(z, s) + \sum_{n=1}^{\infty} C_N(n) \sum_{m=1}^{\infty} \frac{1}{m} (q_z^{mn} + \bar{q}_z^{mn}) + O(s-1). \end{aligned}$$

Combining this with (3.6) and Proposition 3.3, we obtain

$$\begin{aligned} &\lim_{s \rightarrow 1} \left(\mathcal{E}(N, z, s) - \varphi(N) \zeta^*(2s-1) \right) \\ &= -\frac{\varphi(N)}{2} \log y + \frac{N\varphi(N) \prod_{p|N} (1+p^{-1}) \pi y}{6} - \sum_{n=1}^{\infty} \log |1 - q_z^n|^2 \\ &= -\frac{1}{12} \log (y^{6\varphi(N)} | \Delta_N(z) |), \end{aligned}$$

as claimed. The second one follows from this identity immediately by applying w_N on both sides.

Proof of Theorem 1.6: The first identity is just restatement of Corollary 2.3. For the second identity, we have by Theorems 1.5, 1.4

and Corollary 2.3

$$\begin{aligned}
& -\frac{1}{12}I(\tau, \log |\Delta_N(z)y^{6\varphi(N)}|) \\
& = \lim_{s \rightarrow 1} \left(I(\tau, \mathcal{E}(N, z, s)) - I(\tau, \varphi(N)\zeta^*(2s-1)) \right) \\
& = \lim_{s \rightarrow 1} \left(\zeta^*(s)\mathcal{E}_L(\tau, s) - 2\zeta^*(2s-1)\mathcal{E}_L(\tau, 1) \right).
\end{aligned}$$

Now the second identity for $\log \|\Delta_N(z)\|$ follows from elementary calculation of the Laurent expansion (just first two terms) of the functions in the above expression. We leave the detail to the reader.

Proposition 3.5. (1) *The generalized Delta function $\Delta_N(z)$ of level N vanishes at the cusp ∞ with vanishing order $N\varphi(N) \prod_{p|N}(1+p^{-1})$, and does not vanish at other cusps.*

(2)

$$(3.8) \quad \Delta_N^0(z) = \Delta_N(z)|_{w_N} = C_N \prod_{t|N} \Delta(tz)^{a(\frac{N}{t})} \in M_k(N)$$

has vanishing order $\varphi(N)N \prod_{p|N}(1+p^{-1})$ at the cusp P_0 and does not vanish at other cusps. Here C_N is the constant given in Proposition 3.4.

Proof. This proposition is clear at cusp $P_{Q/N}$ with $Q \parallel N$ by Proposition 3.4. In particular, it is true when N is square free, which is all what we needed in Part II. The general case follows from the Kronecker limit formula at the cusp P . Write

$$N^{2s}\pi^{-s}\Gamma(s)\zeta^{(N)}(2s) = A + B(s-1) + o(s-1),$$

and $\alpha = \frac{\varphi(N)}{A}$. According to [Go, (21)], for a cusp P , there is some $\sigma = \sigma_P \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma(P_\infty) = P$, and

$$\begin{aligned}
& \lim_{s \rightarrow 1} \left(E(N, \sigma z, s) - \frac{\alpha}{2(s-1)} \right) \\
& = \beta_P - \frac{\alpha}{2} \log y + y\delta_{P, P_\infty} + \sum_{m>1} (\phi_{P,m}q_z^m + \overline{\phi_{P,m}q_z^m}),
\end{aligned}$$

for some constant β_P . Here δ_{P, P_∞} is the Kronecker δ -symbol. So simple calculation gives for $P \neq P_\infty$

$$\begin{aligned}
& \lim_{s \rightarrow 1} \left(\mathcal{E}(N, \sigma z, s) - \varphi(N)\zeta^*(2s-1) \right) \\
& = \gamma_P - \frac{\varphi(N)}{2} \log y + A \sum_{m>1} (\phi_{P,m}q_z^m + \overline{\phi_{P,m}q_z^m}),
\end{aligned}$$

for some constant γ_P . One has thus by Theorem 1.5

$$\log \left(y^{6\varphi(N)} \mid \Delta_N(\sigma(z)) \mid \right) = -12\gamma_P + 6\varphi(N) \log y - 12A \sum_{m>1} (\phi_{P,m} q_z^m + \overline{\phi_{P,m} q_z^m}).$$

Equivalently,

$$\log |\Delta_N(\sigma(z))| = -12\gamma_P - 12A \sum_{m>1} (\phi_{P,m} q_z^m + \overline{\phi_{P,m} q_z^m}),$$

which goes to $-12\gamma_P$ when $y \rightarrow \infty$. So $\Delta_N(z)$ does not vanish at the cusp $P = \sigma(P_\infty)$. \square

Recall that the Eisenstein series $E(N, z, s)$ has the Fourier expansion

$$E(N, z, s) = \sum_{n \in \mathbb{Z}} c_n(y, s) e(nx),$$

where the constant term has the form

$$c_0(y, s) = y^s + \Phi(s) y^{1-s},$$

with

$$(3.9) \quad \Phi(s) = \frac{\varphi(N) \pi^{\frac{1}{2}} \zeta(2s-1) \Gamma(s - \frac{1}{2})}{N^{2s} \zeta(N)(2s)}.$$

Simple calculation gives the following lemma, which will be used in the proof of Theorem 6.9.

Lemma 3.6. *Write*

$$\Phi(s) = \frac{C_{-1}}{s-1} + C_0 + O(s-1).$$

Then

$$C_{-1} = \text{Res}_{s=1} \Phi(s) = \frac{3}{\pi r},$$

$$C_0 = -\frac{6}{\pi r} \left(\log 4\pi - 1 + 12\zeta'(-1) + \sum_{p|N} \frac{p^2}{p^2-1} \log p \right),$$

where $r = [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1})$.

We remark that C_0 is the so-called scattering constant C_{P_∞, P_∞} in [Kü1].

Part 2. Arithmetic intersection and derivative of Eisenstein series

In this part, we will focus on the arithmetic intersection on the modular curve $X_0(N)$ and prove Theorems 1.3 and 1.1. We will assume from now on that N is square free.

4. METRIZED LINE BUNDLES WITH LOG SINGULARITY AND ARITHMETIC DIVISORS WITH LOG-LOG SINGULARITIES

The Gillet-Soule height pairing (see [So]) has been extended to arithmetic divisors with log-log singularities or equivalently metrized bundles with log singularities ([BKK], [Kü1], [Kü2]). It is also extended to arithmetic divisors with L_1^2 -Green functions ([Bo]). In this paper, we will use Kühn's set-up in [Kü1], which is most convenient in our situation. Actually, for simplicity, we use a stronger condition which is easier to state and enough for our purpose.

Let \mathcal{X} be a regular and proper stack over \mathbb{Z} of dimension 2 (called arithmetic surface), and denote $X = \mathcal{X}(\mathbb{C})$. For a finite subset $S = \{S_1, \dots, S_r\}$ of X , let $Y = X - S$ be its complement. For $\epsilon > 0$, let $B_\epsilon(S_j)$ be the open disc of radius ϵ centered at S_j , and $X_\epsilon = X - \bigcup_j B_\epsilon(S_j)$. Let t_j be a local parameter at S_j . A metrized line bundle $\widehat{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ with log singularity (with respect to S) is a line bundle \mathcal{L} over \mathcal{X} together with a metric $\|\cdot\|$ on $\mathcal{L}_\infty = \mathcal{L}(\mathbb{C})$ satisfying the following conditions:

- (1) $\|\cdot\|$ is a smooth hermitian metric on \mathcal{L}_∞ when restricting to Y .
- (2) For each $S_j \in S$ and a (non-trivial) section s of \mathcal{L} , there is a real number α_j and a positive smooth function φ on $B_\epsilon(S_j)$ such that

$$\|s(t_j)\| = (-\log |t_j|^2)^{\alpha_j} |t_j|^{\text{ord}_{S_j}(s)} \varphi(t_j)$$

hold for all $t_j \in B_\epsilon(S_j) - \{0\}$ (here $t_j = 0$ corresponds to S_j).

Notice that $\widehat{\mathcal{L}}$ with log singularity is a regular metrized line bundle if and only if all $\alpha_j = 0$. We will denote $\widehat{\text{Pic}}_{\mathbb{R}}(\mathcal{X}, S)$ for the group of metrized line bundles with log singularity (with respect to S) with \mathbb{R} -coefficients (i.e. allowing formally $\widehat{\mathcal{L}}^c$ with $c \in \mathbb{R}$).

An pair $\widehat{\mathcal{Z}} = (\mathcal{Z}, g)$ is called an arithmetic divisor with log-log-singularity (along S) if \mathcal{Z} is a divisor of \mathcal{X} , and g is a smooth function

away from $Z \cup S$ ($Z = \mathcal{Z}(\mathbb{C})$), and satisfying the following conditions:

$$\begin{aligned} dd^c g + \delta_Z &= [\omega], \\ g(t_j) &= -2\alpha_j \log \log \left(\frac{1}{|t_j|^2} \right) - 2\beta_j \log |t_j| - 2\psi_j(t_j) \quad \text{near } S_j, \end{aligned}$$

for some smooth function ψ_j and some $(1, 1)$ -form ω which is smooth away from S . Let $\widehat{\mathcal{L}}$ be the metrized line bundle associated to $\widehat{\mathcal{Z}}$ with canonical section s with $-\log \|s\|^2 = g$, then $\widehat{\mathcal{Z}}$ is of log-log-singularity if and only if $\widehat{\mathcal{L}}$ has log-growth and

$$(4.1) \quad \alpha_j(g) = \alpha_j(s), \quad \beta_j(g) = \text{ord}_{S_j}(s), \quad \psi_j(t_j) = \log \varphi(t_j).$$

We define $\widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}, S)$ be the quotient of the \mathbb{R} -linear combination of the arithmetic divisors of \mathcal{X} with log-log growth along S by \mathbb{R} -linear combinations of the principal arithmetic divisors with log-log growth along S . One has $\widehat{\text{Pic}}_{\mathbb{R}}(\mathcal{X}, S) \cong \widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}, S)$. The following is a [Kü1, Proposition 1.4].

Proposition 4.1. *There is an extension of the Gillet-Soule height pairing to*

$$\widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}, S) \times \widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}, S) \rightarrow \mathbb{R}$$

such that if \mathcal{Z}_1 and \mathcal{Z}_2 are divisors intersect properly, then

$$\langle (\mathcal{Z}_1, g_1), (\mathcal{Z}_2, g_2) \rangle = (\mathcal{Z}_1 \cdot \mathcal{Z}_2)_{\text{fin}} + \frac{1}{2} g_1 * g_2$$

where the star product is defined to be

$$\begin{aligned} g_1 * g_2 &= g_1(Z_2 - \sum_j \text{ord}_{S_j}(Z_2) S_j) + 2 \sum_j \text{ord}_{S_j} Z_2 (\alpha_{\mathcal{L}_{1,j}} - \psi_{1,j}(0)) \\ &\quad - \lim_{\epsilon \rightarrow 0} \left(2 \sum_j (\text{ord}_{S_j} Z_2) \alpha_{\mathcal{L}_{1,j}} \log(-2 \log \epsilon) - \int_{X_\epsilon} g_2 \omega_1 \right). \end{aligned}$$

Here $Z_i = \mathcal{Z}_i(\mathbb{C})$, $\widehat{\mathcal{L}}_i$ is the associated metrized line bundle with the canonical section s_i . $\alpha_{\mathcal{L}_{i,j}}$ and $\psi_{i,j}$ are associated to g_i and cusp S_j . Finally, ω_i is the $(1, 1)$ -form associate to g_i via the following equation

$$dd^c g_i + \delta_{Z_i} = [\omega_i].$$

We remark that the pairing is also symmetric. In particular, one has for a smooth function f on X

$$(4.2) \quad \langle (\mathcal{Z}, g), (0, f) \rangle = \frac{1}{2} \int_X f \omega.$$

We define the degree map

$$(4.3) \quad \deg : \widehat{\mathrm{CH}}_{\mathbb{R}}^1(\mathcal{X}, S) \rightarrow \mathbb{R}, \quad \deg(\mathcal{Z}, g) = \int_X \omega = \langle (\mathcal{Z}, g), (0, 2) \rangle.$$

It is just $\deg Z$ when g is a green function of $Z = \mathcal{Z}(\mathbb{C})$ without log-log singularity.

We will denote $\widehat{\mathrm{CH}}_{\mathbb{R}}^1(\mathcal{X}) = \widehat{\mathrm{CH}}_{\mathbb{R}}^1(\mathcal{X}, \text{empty})$ for the usual arithmetic Gillet-Soule Chow group with real coefficients.

5. KUDLA'S GREEN FUNCTION

Let $V = \{w \in M_2(\mathbb{Q}) : \mathrm{tr}(w) = 0\}$ be the quadratic space with quadratic form $Q(w) = N \det w$, and let \mathbb{D} be the associated Hermitian symmetric domain of positive lines in $V_{\mathbb{R}}$ as in Section 2. Recall that $\mathrm{SL}_2 = \mathrm{Spin}(V)$ acts on \mathbb{D} by conjugation, and \mathbb{D} can be identified with \mathbb{H} (Lemma 2.1) via

$$(5.1) \quad w(z) = \frac{1}{\sqrt{N}y} \begin{pmatrix} -x & z\bar{z} \\ -1 & x \end{pmatrix}, \quad z = x + iy \in \mathbb{H}.$$

Let L be an even integral lattice with dual lattice L^{\sharp} . Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index which fixes L and acts on L^{\sharp}/L trivially. We denote $\bar{\Gamma} = \Gamma/(\Gamma \cap \{\pm 1\})$. For each pair $(n, \mu) \in \mathbb{Q} \times L^{\sharp}/L$ with $n > 0$, $Q(\mu) \equiv n \pmod{1}$, let $Z(n, \mu)$ be the associated Heegner divisor given by

$$Z(n, \mu) = \Gamma \backslash \{\mathbb{R}w : w \in L_{\mu}[n]\}.$$

Kudla defined a nice Green function for $Z(n, \mu)$ in his seminal work [Ku1], which we now briefly review. The purpose of this section is to understand its behavior at the cusps.

For $r > 0$ and $s \in \mathbb{R}$, let

$$(5.2) \quad \beta_s(r) = \int_1^{\infty} e^{-rt} t^{-s} dt$$

and

$$(5.3) \quad \xi(w, z) = \beta_1(2\pi R(w, z)),$$

be Kudla's ξ -function. For $\mu \in L^{\sharp}/L$, $n \in Q(\mu) + \mathbb{Z}$ and $v \in \mathbb{R}_{>0}$, define

$$(5.4) \quad \Xi(n, \mu, v)(z) = \sum_{0 \neq w \in L_{\mu}[n]} \xi(v^{\frac{1}{2}}w, z).$$

Then Kudla has proved on $Y_0(N)$ ([Ku1]) that $\Xi(n, \mu, v)$ is a Green function for $Z(n, \mu)$ and satisfies the following current equation: one

has the following identity when both sides are well-defined

$$dd^c \Xi(n, \mu, v) + \delta_{Z(n, \mu)} = [\omega(n, \mu, v)]$$

when $n > 0$ and is smooth when $n < 0$. The purpose of this section is understand its behavior at cusps.

Let $\text{Iso}(V)$ be the set of isotropic non-zero vectors of V , i.e., $0 \neq \ell \in V$ with $Q(\ell) = 0$. Given $\ell = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Iso}(V)$, let $P_\ell = \frac{a}{c}$ be the associated cusp, which depends only on the isotropic line $\mathbb{Q}\ell$. Two isotropic lines give the same cusp in $\Gamma \backslash \mathbb{H}$ if and only if there is $\gamma \in \Gamma$ such that $\mathbb{Q}\gamma\ell_1 = \mathbb{Q}\ell_2$.

Let $\ell_\infty = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \text{Iso}(V)$ and let $P_\infty = \infty$ be its associated cusp at ∞ . In general, for an isotropic element ℓ , there exists $\sigma_\ell \in \text{SL}_2(\mathbb{Z})$ such that $\mathbb{Q}\sigma_\ell\ell_\infty = \mathbb{Q}\ell$. Then

$$\sigma_\ell^{-1} \Gamma_\ell \sigma_\ell = \{ \pm \begin{pmatrix} 1 & m\kappa_\ell \\ 0 & 1 \end{pmatrix}, m \in \mathbb{Z} \},$$

where Γ_ℓ is the stabilizer of ℓ and $\kappa_\ell > 0$ is the classical width of the associated cusp P_ℓ , and q_ℓ is a local parameter at the cusp P_ℓ . On the other hand, there is another positive number $\beta_\ell > 0$, depending on L and the cusp P_ℓ , such that $\begin{pmatrix} 0 & \beta_\ell \\ 0 & 0 \end{pmatrix}$ is a primitive element in $\mathbb{Q}\ell_\infty \cap \sigma_\ell^{-1}L$. We denote $\varepsilon_\ell = \frac{\kappa_\ell}{\beta_\ell}$ and call it Funke constant at cusp P_ℓ although Funke called it width at P_ℓ in [Fu, Section 3]. We will simply denote $\kappa = \kappa_\infty$.

The main purpose of this section is to prove the following technical theorem.

Theorem 5.1. *Let the notation be as above. Let $0 \neq \ell \in \text{Iso}(V)$ be an isotropic vector and P_ℓ be the associated cusp.*

- (1) *When $D = -4nN$ is not a square, $\Xi(n, \mu, v)$ is smooth and of exponential decay at the cusp P_ℓ .*
- (2) *When $D = -4nN > 0$ is a square. Then $\Xi(n, \mu, v)$ has log singularity at the cusp P_ℓ with*

$$\Xi(n, \mu, v) = -g(n, \mu, v, P_\ell)(\log |q_\ell|^2) - 2\psi_\ell(n, \mu, v; q_\ell).$$

Here q_ℓ is a local parameter at the cusp P_ℓ ,

$$\alpha_\Gamma(n, \mu, P_\ell) = \sum_{w \in L_\mu[n] \bmod \Gamma} \delta_w,$$

where $0 \leq \delta_w \leq 2$ is the number of isotropic lines $\mathbb{Q}\ell_w \in \text{Iso}(V)$ which is perpendicular to w and belongs to the same cusp as ℓ , and

$$g(n, \mu, v, P_\ell) = \frac{1}{8\pi\sqrt{-nv}} \beta_{3/2}(-4nv\pi) \alpha_\Gamma(n, \mu, P_\ell).$$

Finally, $\psi_\ell(n, \mu, v; q_\ell)$ is a smooth function of q_ℓ (as two real variables q_ℓ and \bar{q}_ℓ) and

$$\lim_{q_\ell \rightarrow 0} \psi_\ell(n, \mu, v; q_\ell) = 0.$$

(3) When $D = 0$, one has

$$\begin{aligned} \Xi(0, \mu, v) &= -g(0, \mu, v, P_\ell)(\log |q_\ell|^2) - 2 \log(-\log |q_\ell|^2) \\ &\quad - 2\psi_\ell(0, \mu, v; q_\ell), \end{aligned}$$

where q_ℓ is the local parameter at P_ℓ with respect to the classical width κ_ℓ , $g(0, \mu, v, P_\ell) = \frac{\varepsilon_\ell}{2\pi\sqrt{vN}}$. Here ε_ℓ is the Funke constant of ℓ . Finally, $\psi_\ell(0, \mu, v; q_\ell)$ is a smooth function of q_ℓ (as two real variables q_ℓ and \bar{q}_ℓ) and

$$\lim_{q_\ell \rightarrow 0} \psi_\ell(0, \mu, v; q_\ell) = \begin{cases} \log \frac{\varepsilon_\ell}{4\pi\sqrt{Nv}} - \frac{1}{2}f(0) & \text{if } \mu \in L, \\ \frac{1}{2} \log \frac{\varepsilon_\ell^2}{4Nv\pi^3} + \frac{\gamma_1(0)}{2} - \sum_{n=1}^{\infty} \frac{\cos(\frac{2\pi n\mu_\ell}{\beta_\ell})}{n} & \text{if } \mu \notin L. \end{cases}$$

Here $f(0) = \gamma - \log(4\pi)$ is defined in Lemma 5.2,

$$\gamma_1(0) = \int_1^\infty e^{-y} \frac{dy}{y} + \int_0^1 \frac{e^{-y} - 1}{y} dy$$

and

$$\sigma_\ell^{-1} L_\mu \cap \mathbb{Q}\ell = \left\{ \begin{pmatrix} 0 & \mu_\ell + m\beta_\ell \\ 0 & 0 \end{pmatrix} : m \in \mathbb{Z} \right\}.$$

The proof is long and technical and will occupy the next few subsections. A reader can safely skip the detail to Section 6.

5.1. Two lemmas.

Lemma 5.2. *Let $a > 0$ and $z = x + iy \in \mathbb{C}$. Then*

(1) *When $z \notin \mathbb{R}$, one has*

$$\sum_{n \in \mathbb{Z}} \beta_1(\pi a^2 |z + n|^2) = \frac{1}{a} \sum_{n \in \mathbb{Z}} e(nx) \int_1^\infty e^{-\pi a^2 y^2 t - \frac{\pi n^2}{a^2 t} t^{-\frac{3}{2}}} dt.$$

(2) *When $z = x \in \mathbb{R} - \mathbb{Z}$, one has*

$$\sum_{n \in \mathbb{Z}} \beta_1(\pi a^2 (x + n)^2) = 2 \sum_{n \in \mathbb{Z}} e(nx) \int_0^{\frac{1}{a}} e^{-\pi n^2 t^2} dt.$$

Moreover, one has near $a = 0$

$$\sum_{n \in \mathbb{Z}} \beta_1(\pi a^2 (x + n)^2) = \frac{2}{a} + f(a, x),$$

for some smooth function $f(a, x)$ near $a = 0$ with

$$f(0, x) = \lim_{a \rightarrow 0} f(a, x) = 2 \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n}.$$

(3) One has

$$\sum_{0 \neq n \in \mathbb{Z}} \beta_1(\pi a^2 n^2) = 2 \int_0^{\frac{1}{a}} \left(\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t^2} - \int_{\mathbb{R}} e^{-\pi x^2 t^2} dx \right) dt.$$

Moreover, One has near $a = 0$

$$\sum_{0 \neq n \in \mathbb{Z}} \beta_1(\pi a^2 n^2) = \frac{2}{a} + 2 \log a + f(a),$$

for some smooth function $f(a)$ near $a = 0$ with

$$f(0) = \lim_{a \rightarrow 0} f(a) = \gamma - \log(4\pi),$$

where γ is the Euler constant.

Proof. Let

$$f(n) = \beta_1(\pi a^2 |z + n|^2) = \beta_1(\pi a^2 y^2 + \pi a^2 (x + n)^2).$$

Then its Fourier transformation is

$$\begin{aligned} \widehat{f}(n) &= \int_{\mathbb{R}} f(\alpha) e(-\alpha n) d\alpha \\ &= \frac{e(nx)}{a} \int_1^{\infty} e^{-\pi a^2 y^2 t - \frac{\pi n^2}{a^2 t}} t^{-\frac{3}{2}} dt. \end{aligned}$$

Now applying the Poisson summation formula, one obtains the formula in (1). When $y = 0$, simple substitution gives part of (2) with $x \notin \mathbb{Z}$. To see the behavior of the sum near $a = 0$, notice that the right-hand side is equal to $\frac{2}{a} + f(a, x)$ with

$$f(a, x) = 2 \sum_{n=1}^{\infty} (e(nx) + e(-nx)) \int_0^{\frac{1}{a}} e^{-\pi n^2 t^2} dt.$$

It is clearly smooth near $a = 0$ if we define

$$f(0, x) = 2 \sum_{n=1}^{\infty} (e(nx) + e(-nx)) \int_0^{\infty} e^{-\pi n^2 t^2} dt = 2 \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n}.$$

To prove (3), take $z = i\epsilon$ in (1), and let ϵ goes to zero, we obtain

$$\sum_{0 \neq n \in \mathbb{Z}} \beta_1(\pi a^2 n^2) = \lim_{\epsilon \rightarrow 0} \left[\frac{1}{a} \sum_{n \in \mathbb{Z}} \int_1^{\infty} e^{-\pi a^2 \epsilon^2 t - \frac{\pi n^2}{a^2 t}} t^{-\frac{3}{2}} dt - \beta_1(\pi a^2 \epsilon^2) \right].$$

By the Fourier inversion formula, one has

$$\frac{1}{a} \int_1^\infty \int_{\mathbb{R}} e^{-\pi a^2 \epsilon^2 t - \frac{\pi x^2}{a^2} t^{-1}} t^{-\frac{3}{2}} dx dt = \beta_1(\pi a^2 \epsilon^2).$$

So

$$\begin{aligned} & \sum_{0 \neq n \in \mathbb{Z}} \beta_1(\pi a^2 n^2) \\ &= \frac{1}{a} \lim_{\epsilon \rightarrow 0} \int_1^\infty e^{-\pi a^2 \epsilon^2 t} \left[\sum_{n \in \mathbb{Z}} e^{-\frac{\pi n^2}{a^2 t}} - \int_{\mathbb{R}} e^{-\frac{\pi x^2}{a^2 t}} dx \right] t^{-\frac{3}{2}} dt \\ &= \frac{2}{a} \int_1^\infty \left[\sum_{n \in \mathbb{Z}} e^{-\frac{\pi n^2}{a^2 t}} - \int_{\mathbb{R}} e^{-\frac{\pi x^2}{a^2 t}} dx \right] t^{-\frac{3}{2}} dt \\ &= 2 \int_0^{\frac{1}{a}} \left(\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t^2} - \int_{\mathbb{R}} e^{-\pi x^2 t^2} dx \right) dt \\ &= \frac{2}{a} - 4 \int_0^{\frac{1}{a}} \int_0^1 e^{-\pi x^2 t^2} dx dt + 4 \sum_{n=1}^\infty \left[\int_0^{\frac{1}{a}} e^{-\pi n^2 t^2} dt - \int_0^{\frac{1}{a}} \int_n^{n+1} e^{-\pi x^2 t^2} dx dt \right] \\ &= \frac{2}{a} - 4g_0(a) + 4 \sum_{n=1}^\infty g_n(a), \end{aligned}$$

with obvious meaning of $g_n(a)$. Here we have used the fact that the integrand in the last integral is negative. The term $\frac{2}{a}$ comes from the term $n = 0$ in the sum. We remark that the formula looks formally like ($z = 0$)

$$\sum_{n \neq 0} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) - \int_{\mathbb{R}} \hat{f}(x) dx.$$

What we did is to regularize the right hand side to make it meaningful.

First,

$$\begin{aligned} g_0(a) &= \int_0^1 \int_0^1 e^{-\pi x^2 t^2} dx dt + \int_1^{\frac{1}{a}} \int_0^1 e^{-\pi x^2 t^2} dx dt \\ &= \int_0^1 \int_0^1 e^{-\pi x^2 t^2} dx dt + \int_1^{\frac{1}{a}} \int_0^\infty e^{-\pi x^2 t^2} dx dt - \int_1^{\frac{1}{a}} \int_1^\infty e^{-\pi x^2 t^2} dx dt \\ &= -\frac{1}{2} \log a + \int_0^1 \int_0^1 e^{-\pi x^2 t^2} dx dt - \int_1^{\frac{1}{a}} \int_1^\infty e^{-\pi x^2 t^2} dx dt. \end{aligned}$$

Using Mathematica, we see

$$\lim_{a \rightarrow 0} (g_0(a) + \frac{1}{2} \log a) = \int_0^1 \int_0^1 e^{-\pi x^2 t^2} dx dt - \int_1^\infty \int_1^\infty e^{-\pi x^2 t^2} dx dt = \frac{1}{4}(\gamma + \log 4\pi).$$

Next, One has

$$\begin{aligned} & \lim_{a \rightarrow 0} \sum_{n=1}^{\infty} \left[\int_0^{\frac{1}{a}} e^{-\pi n^2 t^2} dt - \int_0^{\frac{1}{a}} \int_n^{n+1} e^{-\pi x^2 t^2} dx dt \right] \\ &= \sum_{n=1}^{\infty} \left[\int_0^{\infty} e^{-\pi n^2 t^2} dt - \int_n^{n+1} \int_0^{\infty} e^{-\pi x^2 t^2} dt dx \right] \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \log \frac{n+1}{n} \right) = \frac{1}{2} \gamma. \end{aligned}$$

In summary, one has

$$\sum_{0 \neq n \in \mathbb{Z}} \beta_1(\pi a^2 n^2) = \frac{2}{a} + 2 \log a + f(a),$$

for some smooth function $f(a)$ near $a = 0$ with

$$f(0) = \lim_{a \rightarrow 0} f(a) = -(\gamma + \log(4\pi)) + 2\gamma = \gamma - \log(4\pi).$$

□

Lemma 5.3. *Assume that $D = -4Nn = (2Nm)^2 > 0$ is a square. For any $w = w(m, r) = m \begin{pmatrix} 1 & 2r \\ 0 & -1 \end{pmatrix} \in L_\mu[n]$ with $(w, \ell_\infty) = 0$, define*

$$\Xi_\infty(w, z) = \sum_{\gamma \in \bar{\Gamma}_\infty} \xi(w, \gamma z).$$

Then for any $v > 0$

$$\Xi_\infty(\sqrt{v}w, z) = -(\log |q_\kappa|) \frac{\sqrt{N}}{2\pi\sqrt{D}v} \sum_{n \in \mathbb{Z}} e\left(\frac{n}{\kappa}(x+r)\right) \int_1^\infty e^{-\left(\frac{tDv}{N} + n^2 \frac{N}{tDv} \frac{y^2}{\kappa^2}\right)\pi} \frac{dt}{t^{\frac{3}{2}}},$$

where $q_\kappa = e(z/\kappa)$ is a local parameter of X_Γ at the cusp P_∞ . Moreover, one has near the cusp P_∞ ($q_\kappa = 0$)

$$\Xi_\infty(\sqrt{v}w, z) = -(\log |q_\kappa|^2) \frac{\sqrt{N}}{4\pi\sqrt{D}v} \beta_{\frac{3}{2}}\left(\frac{Dv\pi}{N}\right) + f(\sqrt{v}w, z),$$

where $f(\sqrt{v}w, z)$ is a smooth function of x and y near P_∞ and

$$\lim_{y \rightarrow \infty} f(\sqrt{v}w, z) = 0.$$

Proof. One has $\bar{\Gamma}_\infty = \{ \begin{pmatrix} 1 & \kappa\mathbb{Z} \\ 0 & 1 \end{pmatrix} \}$ and

$$\begin{aligned} R(\sqrt{v}w, \begin{pmatrix} 1 & n\kappa \\ 0 & 1 \end{pmatrix} z) &= \frac{v}{2} (w, w(z + n\kappa))^2 - v(w, w) \\ &= \frac{Dv}{2Ny^2} |z + n\kappa + r|^2. \end{aligned}$$

So one has by Lemma 5.2,

$$\begin{aligned}
\Xi_\infty(\sqrt{v}w, z) &= \sum_{n \in \mathbb{Z}} \beta_1\left(\frac{\pi Dv}{Ny^2} |z + r + n\kappa|^2\right) \\
&= \frac{y\sqrt{N}}{\kappa\sqrt{Dv}} \sum_{n \in \mathbb{Z}} e\left(\frac{n}{\kappa}(x+r)\right) \int_1^\infty e^{-\left(\frac{tD}{N} + n^2 \frac{N}{tD} \frac{y^2}{\kappa^2}\right)\pi} \frac{dt}{t^{\frac{3}{2}}} \\
&= -(\log |q_\kappa|^2) \frac{\sqrt{N}}{4\pi\sqrt{Dv}} \beta_{\frac{3}{2}}\left(\frac{\pi D}{N}\right) + f(\sqrt{v}w, z)
\end{aligned}$$

with

$$f(\sqrt{v}w, z) = -(\log |q_\kappa|^2) \frac{\sqrt{N}}{4\pi\sqrt{Dv}} \sum_{0 \neq n \in \mathbb{Z}} e\left(\frac{n}{\kappa}(x+r)\right) \int_1^\infty e^{-\left(\frac{tDv}{N} + n^2 \frac{N}{tDv} \frac{y^2}{\kappa^2}\right)\pi} \frac{dt}{t^{\frac{3}{2}}}.$$

Since

$$\frac{tDv}{N} + n^2 \frac{N}{tDv} \frac{y^2}{\kappa^2} \geq \frac{2|n|y}{\kappa},$$

one sees for all $n \neq 0$

$$\left| e\left(\frac{n}{\kappa}(x+r)\right) \int_1^\infty e^{-\left(\frac{tDv}{N} + n^2 \frac{N}{tDv} \frac{y^2}{\kappa^2}\right)\pi} \frac{dt}{t^{\frac{3}{2}}} \right| \leq 2e^{-2\frac{|n|y}{\kappa}\pi},$$

and

$$|f(\sqrt{v}w, z)| \leq \frac{4\sqrt{N}y}{\kappa\sqrt{Dv}} \sum_{n=1}^\infty e^{-2\frac{n\pi y}{\kappa}}$$

which is of exponential decay as $y \mapsto \infty$. This proves the lemma. \square

5.2. Proof of Theorem 5.1.

Proof. Now we are ready to start proof of Theorem 5.1. By transformation, we may assume that $\ell = \ell_\infty$ is associated to the cusp P_∞ . Then $q_\ell = q_\kappa$ where κ is the width of the cusp P_∞ as in the above lemma. We divide the proof into three steps: general set-up and the case $D = -4Nn$ is not a square, $D > 0$ being a square, and $D = 0$.

Step 1: Set-up and the case that D is not a square. We write

$$(5.5) \quad \Xi(n, \mu, v) = \sum_{w \in L_\mu[n] \bmod \Gamma} \Xi(\sqrt{v}w, z), \quad \Xi(\sqrt{v}w, z) = \sum_{\gamma \in \bar{\Gamma}_w \backslash \bar{\Gamma}} \xi(\sqrt{v}w, \gamma z).$$

For $w = \begin{pmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{pmatrix} \in L_\mu[n]$, let $\tilde{w} = \begin{pmatrix} w_3 & -w_1 \\ -w_1 & -w_2 \end{pmatrix} = S^{-1}w$ with $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Then \tilde{w} is symmetric. Simple calculation gives for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$(5.6) \quad R(w, \gamma z) = \frac{N}{2y^2} [h_{\tilde{w}}(\gamma, z)]^2 - n,$$

where

$$h_{\tilde{w}}(\gamma, z) = (az+b, cz+d)\tilde{w}\overline{(az+b, cz+d)}^t = Q_{\tilde{w}}(a, c)y^2 + Q_{\tilde{w}}(ax+b, cx+d)$$

is the Hermitian form on $(\mathbb{R}z + \mathbb{R})^2$, and $Q_{\tilde{w}}$ is the quadratic form on \mathbb{R}^2 associated to \tilde{w} . Notice that $\{(az+b, cz+d) : \gamma \in \Gamma\}$ is a subset of a lattice of $(\mathbb{R}z + \mathbb{R})^2$, so for any positive number M

$$\#\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma : |h_{\tilde{w}}(\gamma, z)| \leq M \quad \text{and} \quad 0 < |Q_{\tilde{w}}(a, c)| \leq M\}$$

are finite and of polynomial growth as functions of M . Moreover there is a positive number M_0 such that if $Q_{\tilde{w}}(a, c) \neq 0$ for some $\gamma \in \Gamma$, then $|Q_{\tilde{w}}(a, c)| \geq M_0$. In such a case, we have

$$R(w, \gamma z) \sim \frac{N}{2} Q_{\tilde{w}}(a, c)^2 y^2$$

as $y \rightarrow \infty$. Recall that

$$\beta_1(t) = O(e^{-t}/t)$$

as $t \rightarrow \infty$. Therefore the terms with $Q_{\tilde{w}}(a, c) \neq 0$ in the sum $\Xi(\sqrt{v}w, z)$ goes to zero in an exponential decay fashion. So we have proved the following lemma.

Lemma 5.4. *Let the notation be as above. If there is no $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $Q_{\tilde{w}}(a, c) = 0$, then $\Xi(\sqrt{v}w, z)$ is smooth at the cusp P_∞ and is of exponential decay as $y \rightarrow \infty$.*

When D is not a square, the quadratic form $Q_{\tilde{w}}$ does not represent 0. So $\Xi(n, \mu, v)$ is of exponential decay in this case when $y \rightarrow \infty$. This proves (1).

Step 2: Next, we assume $D = -4Nn > 0$ is a square. In this case, $\bar{\Gamma}_w = 1$

$$0 = Q_{\tilde{w}}(a, c) = w_3 a^2 - 2w_1 ac - w_2 c^2$$

has exactly two integral solutions $(a_i, c_i) \in \mathbb{Z}^2$ such that $\gcd(a_i, c_i) = 1$, $a > 0$ or $a_i = 0, c_i = 1$. So $w^\perp \cap \text{Iso}(V)$ consists exactly two cusps $\mathbb{Q}\ell_{a_i, c_i}$ where $\ell_{a, c} = \begin{pmatrix} ac & -a^2 \\ c^2 & -ac \end{pmatrix}$.

For a fixed solution (a, c) , if there is $\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, then the cusp $P_{\frac{a}{c}}$ (corresponding to $\mathbb{Q}\ell_{a, c}$) is Γ -equivalent to P_∞ : $\gamma_0 P_\infty = P_{\frac{a}{c}}$, and all $\gamma = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \Gamma$ with $Q_{\tilde{w}}(a, c) = 0$ is of the form $\gamma_0 \gamma_1$ with $\gamma_1 \in \Gamma_\infty$.

Therefore the sum related to this solution (a, c) is

$$\begin{aligned}
\sum_{\substack{\gamma = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \bar{\Gamma} \\ Q_{\bar{w}}(a, c) = 0}} \xi(\sqrt{v}w, \gamma z) &= \sum_{\gamma_1 \in \bar{\Gamma}_\infty} \xi(\sqrt{v}\gamma_0^{-1} \cdot w, \gamma_1 z) \\
&= \Xi_\infty(\sqrt{v}\gamma_0^{-1} \cdot w, z) \\
&= -(\log |q_\kappa|^2) \frac{\sqrt{N}}{4\pi\sqrt{Dv}} \beta_{\frac{3}{2}}\left(\frac{Dv\pi}{N}\right) + f(\sqrt{v}\gamma_0^{-1} \cdot w, z)
\end{aligned}$$

by Lemma 5.3. Recall $\lim_{y \rightarrow \infty} f(\sqrt{v}\gamma_0^{-1} \cdot w, z) = 0$ by Lemma 5.3. So we have by Lemma 5.4,

$$\begin{aligned}
\Xi(\sqrt{v}w, z) &= \sum_{\substack{Q_{\bar{w}}(a, c) = 0 \\ \gcd(a, c) = 1 \\ a > 0 \text{ OR } a = 0, c = 1}} \sum_{\gamma = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in \bar{\Gamma}} \xi(\sqrt{v}w, z) + \sum_{\substack{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma} \\ Q_{\bar{w}}(a, c) \neq 0}} \xi(\sqrt{v}w, z) \\
&= -\frac{\delta_w \sqrt{N}}{4\pi\sqrt{Dv}} \beta_{\frac{3}{2}}\left(\frac{Dv\pi}{N}\right) (\log |q_\kappa|^2) + \psi(w, z)
\end{aligned}$$

with $\psi(w, z)$ smooth at the cusp P_∞ and

$$\lim_{y \rightarrow \infty} \psi(w, z) = 0.$$

Combining this with Lemma 5.4, we proved (3) of Theorem 5.1.

Step 3: Finally we assume $n = 0$. Each vector $0 \neq w \in L_\mu[0]$ corresponds to an isotropic line and thus a cusp. We regroup the sum in $\Xi(0, \mu, v)$ in terms of Γ -equivalent cusp classes $[P_r]$, where $r \in \mathbb{Q}$ or ∞ . Let $\ell_r = \begin{pmatrix} r & -r^2 \\ 1 & -r \end{pmatrix}$ be a associated isotropic vector for a rational number r and recall $\ell_\infty = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

$$(5.7) \quad \Xi(0, \mu, v) = \sum_{[P_r]} \sum_{0 \neq w \in L_\mu[0] \cap \mathbb{Q}\ell_r} \Xi(\sqrt{v}w, z).$$

Consider first the sum $[P_\infty]$ part. Let

$$L_\mu[0] \cap \mathbb{Q}\ell_\infty = \{w_m = \begin{pmatrix} 0 & \mu_\infty + m\beta_\infty \\ 0 & 0 \end{pmatrix} \neq 0 : m \in \mathbb{Z}\},$$

where $\beta_\infty = \beta_{\ell_\infty}$ is the constant defined at the beginning of this section and $\mu_\infty \in \mathbb{Q}$. Notice that two different w_m s are not Γ -equivalent, and $\Gamma_{w_m} = \Gamma_\infty$. Simple calculation gives

$$\begin{aligned}
\Xi(\sqrt{v}w_m, z) &= \sum_{\gamma \in \bar{\Gamma}_\infty \setminus \bar{\Gamma}} \beta_1\left(\frac{\pi N v |cz + d|^4 (m\beta_\infty + \mu_\infty)^2}{y^2}\right) \\
&= \beta_1\left(\frac{\pi N v (m\beta_\infty + \mu_\infty)^2}{y^2}\right) + \sum_{\gamma \in \bar{\Gamma}_\infty \setminus \bar{\Gamma}, c > 0} \beta_1\left(\frac{\pi N v |cz + d|^4 (m\beta_\infty + \mu_\infty)^2}{y^2}\right).
\end{aligned}$$

When $\mu_\infty \notin \beta_\infty \mathbb{Z}$ (i.e., $\mu \notin L$), one has by Lemma 5.2

$$\begin{aligned} & \sum_{0 \neq w \in L_\mu[0] \cap \mathbb{Q}\ell_\infty} \Xi(\sqrt{v}w, z) \\ &= \sum_{0 \neq m \in \mathbb{Z}} \beta_1\left(\frac{\pi N v (m\beta_\infty + \mu_\infty)^2}{y^2}\right) + e(\mu, z) \\ &= -\beta_1\left(\frac{\pi N v \mu_\infty^2}{y^2}\right) + \frac{2y}{\beta_\infty \sqrt{Nv}} + f\left(\frac{\beta_\infty \sqrt{Nv}}{y}, \frac{\mu_\infty}{\beta_\infty}\right) + e(\mu, z). \end{aligned}$$

Here

$$e(\mu, z) = \sum_{0 \neq m \in \mathbb{Z}} \sum_{\gamma \in \bar{\Gamma}_\infty \setminus \bar{\Gamma}, c > 0} \beta_1\left(\frac{\pi N v |cz + d|^4 (m\beta_\infty + \mu_\infty)^2}{y^2}\right).$$

Recall that near $t = 0$

$$\beta_1(t) = -\log t + \gamma_1(t)$$

with

$$\gamma_1(t) = \int_1^\infty e^{-y} \frac{dy}{y} + \int_t^1 \frac{e^{-y} - 1}{y} dy.$$

So we have for $\mu \notin L$ (recall $y = -\frac{\kappa}{2\pi} \log |q_\kappa|$)

$$(5.8) \quad \sum_{0 \neq w \in L_\mu[0] \cap \mathbb{Q}\ell_\infty} \Xi(\sqrt{v}w, z) = -\log |q_\kappa|^2 \frac{\varepsilon_\infty}{\pi \sqrt{Nv}} - 2 \log(-\log |q_\kappa|^2) + \psi(\mu, z),$$

where

$$\psi(\mu, z) = -\log \frac{\varepsilon_\infty^2}{4Nv\pi^3} - \gamma_1\left(\frac{\pi N v \beta_\infty^2}{y^2}\right) + f\left(\frac{\beta_\infty \sqrt{Nv}}{y}, \frac{\mu_\infty}{\beta_\infty}\right) + e(\mu, z).$$

It is easy to see that every term in the sum of $e(\mu, z)$ is uniformly of exponential decay (with respect to $c, d, m \in \mathbb{Z}, c > 0, m \neq 0$) as y goes to infinity. So $e(\mu, z)$ is of exponential decay as y goes to infinity. This implies

$$(5.9) \quad \lim_{y \rightarrow \infty} \psi(\mu, z) = -\log \frac{\varepsilon_\infty^2}{4Nv\pi^3} - \gamma_1(0) + 2 \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2\pi n \mu_\infty}{\beta_\infty}\right)}{n}.$$

For $\mu \in L$ (i.e., $\mu = 0$ in $L^\# / L$), one has

$$\begin{aligned} \sum_{0 \neq w \in L[0] \cap \mathbb{Q}\ell_\infty} \Xi(\sqrt{v}w, z) &= \frac{2y}{\beta_\infty \sqrt{Nv}} + 2 \log \frac{\sqrt{Nv} \beta_\infty}{y} + f\left(\frac{\sqrt{Nv} \beta_\infty}{y}\right) + e(0, z) \\ &= -\frac{\varepsilon_\infty}{2\pi \sqrt{Nv}} \log |q_{\ell_\infty}|^2 - 2 \log(-\log |q_{\ell_\infty}|^2) + \psi(0, z), \end{aligned}$$

with

$$\psi(0, z) = 2 \log \frac{4\pi\sqrt{Nv}}{\varepsilon_\infty} + f\left(\frac{\sqrt{Nv}\beta_\infty}{y}\right) + e(0, z).$$

So one has

$$(5.10) \quad \lim_{y \rightarrow \infty} \psi(0, z) = 2 \log \frac{4\pi\sqrt{Nv}}{\varepsilon_\infty} + f(0),$$

as $e(0, z)$ is of exponential decay as y goes to the infinity.

Now look at the sum of $[P_r]$ part, where P_r is not Γ -equivalent to P_∞ .

This implies that there is no $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $\gamma(\infty) = \frac{a}{c} = r$.

For $w = m \begin{pmatrix} r & -r^2 \\ 1 & -r \end{pmatrix} \in L_\mu[0] \cap \mathbb{Q}\ell_r$ so that $\tilde{w} = S^{-1}w = m \begin{pmatrix} 1 & -r \\ -r & r^2 \end{pmatrix}$. For

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, one has

$$R(w, \gamma z) = \frac{1}{2}(w, w(\gamma z))^2 = \frac{Nm^2 |(a - rc)z + (b - rd)|^4}{2y^2} \sim \frac{m^2 N}{2}(a - rc)^4 y^2$$

as $y \rightarrow \infty$ as $a - rc \neq 0$ for all $\gamma \in \Gamma$. So

$$\Xi(\sqrt{v}w, z) = \sum_{\gamma \in \bar{\Gamma}_w \setminus \bar{\Gamma}} \beta_1(2\pi R(w, \gamma z))$$

is smooth and of exponential decay at the cusp P_∞ . Putting everything together, we obtain the result for $\Xi(0, \mu, v)$ at the cusp P_∞ . This finally proves Theorem 5.1. \square

Corollary 5.5. *Let the notation and assumption be as in Theorem 5.1 and let $D = -4nN$. Then $\Xi(n, \mu, v)$ is a Green function for $Z(n, \mu, v)^{\text{Naive}}$ in the usual Gillet-Soule sense for $n \neq 0$ and with (at most) log-log singularity when $n = 0$, and*

$$dd^c \Xi(n, \mu, v) + \delta_{Z(n, \mu, v)^{\text{Naive}}} = [\omega(n, \mu, v)].$$

Here $\omega(n, \mu, v)$ is the differential defined in (2.8)

$$Z(n, \mu, v)^{\text{Naive}} = \begin{cases} Z(n, \mu) & \text{if } D < 0, \\ \sum_{P_\ell \text{ cusps}} g(n, \mu, v, P_\ell) P_\ell & \text{if } D \geq 0 \text{ is a square,} \\ 0 & \text{if otherwise.} \end{cases}$$

Proof. Away from the singularity divisor $Z(n, \mu, v)^{\text{Naive}}$, one has by [Ku1, Proposition 11.1]

$$dd^c \Xi(n, \mu, v) = \omega(n, \mu, v).$$

Near the cusps, it is given by Theorem 5.1, and we leave the detail to the reader following the idea in [Ku1, Proposition 11.1]. \square

6. MODULAR CURVE $\mathcal{X}_0(N)$ AND THE MAIN THEOREM

From now on, we focus on the specific lattice L given in Section 2 and $\Gamma = \Gamma_0(N)$. So our modular curve is $X_0(N) = Y_0(N) \cup S$ the cusp set $S = \{P_{\frac{1}{M}} : M|N\}$ with $P_{\frac{1}{M}}$ is the cusp associated to $\frac{1}{M}$ (as N is square free). Let

$$\ell_{\frac{1}{M}} = \begin{pmatrix} -M & 1 \\ -M^2 & M \end{pmatrix}$$

be an associated isotropic element.

6.1. Some numerical results on Kudla Green functions.

Lemma 6.1. *The Funke constant for $P_{\frac{1}{M}}$ is $\varepsilon_{\frac{1}{M}} = N$, independent of the choices of the cusps.*

Proof. Take $\sigma_M = \begin{pmatrix} 1 & 0 \\ M & 1 \end{pmatrix}$. Then $\sigma_M \ell_{\infty} = \ell_{\frac{1}{M}}$, and

$$\sigma_M^{-1} L \cap \mathbb{Q} \ell_{\infty} = \begin{pmatrix} 0 & \frac{1}{M}\mathbb{Z} \\ 0 & 0 \end{pmatrix}.$$

So we have $\beta_{\frac{1}{M}} = \frac{1}{M}$. Next, We know that

$$(6.1) \quad \sigma_M^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \sigma_M = \begin{pmatrix} 1+Mx & x \\ -M^2x & 1-Mx \end{pmatrix} \in \Gamma_0(N)$$

if and only if $x \in \frac{N}{M}\mathbb{Z}$. This implies $\kappa_{\frac{1}{M}} = \frac{N}{M}$. So $\varepsilon_{\frac{1}{M}} = N$. □

Lemma 6.2. *When $D = -4nN > 0$ is a square, one has for every cusp P_{ℓ}*

$$\alpha_{\Gamma_0(N)}(n, \mu, P_{\ell}) = \begin{cases} \sqrt{D}, & \text{if } 2\mu \notin L, \\ 2\sqrt{D}, & \text{if } 2\mu \in L. \end{cases}$$

Proof. We will drop the subscript $\Gamma_0(N)$ in the proof. We first assume $P_{\ell} = P_{\infty}$. Recall

$$\alpha(n, \mu, P_{\infty}) = \sum_{w \in L_{\mu}[n] \bmod \Gamma_0(N)} \delta_w,$$

where δ_w is the number of the isotropic lines $\mathbb{Q}\ell$ which is perpendicular to w and whose associated cusp is $\Gamma_0(N)$ -equivalent to P_{∞} . By changing w by its $\Gamma_0(N)$ -equivalent element if necessary we may and will assume $(w, \ell_{\infty}) = 0$ (for $\delta_w \neq 0$). This implies

$$w = w(a, b) = \begin{pmatrix} \frac{a}{2N} & \frac{b}{N} \\ 0 & -\frac{a}{2N} \end{pmatrix}$$

with

$$(6.2) \quad a^2 = D, \quad a \equiv r \pmod{2N}.$$

So

$$(6.3) \quad w(a, b)^\perp \cap \text{Iso}(V) = \mathbb{Q}\ell_\infty \cup \mathbb{Q}\ell(a, b), \quad \ell(a, b) = \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}.$$

On the other hand, it is straightforward to check that $w(a, b_1)$ is $\Gamma_0(N)$ -equivalent to $w(a, b_2)$ if and only if $b_1 \equiv b_2 \pmod{a}$. Therefore, we only need to consider these $w(a, b)$ with a satisfying (6.2) and $b \pmod{a}$. There are at most $2|a|$ of them.

Now divide the proof into two cases: $N \nmid r$ (i.e., $2\mu_r \notin L$) and $N|r$ (i.e., $2\mu_r \in L$).

Assume first that $N \nmid r$. Then (6.2) has a unique solution a , and for this a , the cusp $P_{-\frac{b}{a}} = P_{\ell(a, b)}$ is not $\Gamma_0(N)$ -equivalent to P_∞ . So $\delta_w = 1$ for each $w(a, b)$. Therefore we have

$$\alpha(n, \mu, P_\infty) = |a| = \sqrt{D}$$

in this case.

Next we assume $N|r$. In this case (6.2) has two solutions $a = \sqrt{D}$ and $-a$. One has also $N|a$. It is not hard to verify via calculation that $w(a, b)$ and $w(-a, b')$ are $\Gamma_0(N)$ -equivalent if and only if $a_2 = \gcd(a, b) = \gcd(a, b')$ has the following properties: $a = Na_2z$ and $b = a_2w$ with $\gcd(Nz, w) = 1$, and $b' = a_2x$ for some x with $xw - Nyx = 1$ for some integer y . Moreover, in such a case, $b' \pmod{a}$ is uniquely determined by $b \pmod{a}$.

Write $a = Na_1$ and $(a, b) = a_2$ with $b = a_2w$.

Subcase 1: We first assume $a_2|a_1$. In this case, we can write $a_1 = a_2z$ and thus $a = a_2Nz$ with $(w, Nz) = 1$. So $\delta_{w(\pm a, b)} = 2$. On the other hand, $w(\epsilon a, b)$ is $\Gamma_0(N)$ -equivalent to $\Gamma(-\epsilon a, bx)$ with $xw - Nyx = 1$ for some $x, y \in \mathbb{Z}$. So the four pairs $(\pm a, b)$ and $(\pm a, bx)$ contribute 4 to the sum of δ_w .

Subcase 2: Next we assume $a_2 \nmid a_1$. This means $\gcd(a_2, N) > 1$. So the cusp $P_{\frac{b}{\pm a}} = P_{\frac{a_2z}{\pm Na_1}}$ is not $\Gamma_0(N)$ -equivalent to the cusp P_∞ . This implies $\delta_{w(\pm a, b)} = 1$. On the other hand, for such a pair $(\epsilon a, b)$, $w(\epsilon a, b)$ is not $\Gamma_0(N)$ -equivalent to any other $w(\pm a, b')$.

Combining the two subcases, we see that

$$\alpha(n, \mu, P_\infty) = 2|a|$$

in this case. This proves the lemma for the cusp P_∞ .

Next, we show that $\alpha_\Gamma(n, \mu, P_{\frac{1}{M}})$ does not depend on the cusp $P_{\frac{1}{M}}$ in the following sense.

$$(6.4) \quad \alpha_\Gamma(n, \mu, P_{\frac{1}{M}}) = \alpha_\Gamma(n, W_Q \mu W_Q^{-1}, P_\infty),$$

where $Q = \frac{M}{N}$, and W_Q is the associated Atkin-Lehner involution defined as follows. Since $(M, Q) = 1$, there exist $\alpha, \beta \in \mathbb{Z}$ with

$\alpha Q - M\beta = 1$, so $\begin{pmatrix} 1 & \beta \\ M & Q_\alpha \end{pmatrix} \in \Gamma_0(M)$. Let

$$W_Q = \begin{pmatrix} 1 & \beta \\ M & Q_\alpha \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Q & \beta \\ N & Q_\alpha \end{pmatrix}$$

be the associated Atkin-Lehner operator. Then one has

$$W_Q \Gamma_0(N) W_Q^{-1} = \Gamma_0(N).$$

It is easy to verify

$$W_Q L_\mu[n] W_Q^{-1} = L_{W_Q \mu W_Q^{-1}}[n], \quad W_Q \ell_\infty W_Q^{-1} = \begin{pmatrix} Q_\alpha & -\beta \\ M_\alpha & -M\beta \end{pmatrix} = \ell'.$$

Notice that $P_{\ell'} = P_{\frac{1}{M}}$. So there is a bijective map

$$\begin{aligned} L_\mu[n] \bigcap \ell'^{\perp} &\longleftrightarrow L_{W_Q \mu W_Q^{-1}}[n] \bigcap \ell_\infty^{\perp}, \\ w &\longleftrightarrow W_Q^{-1} w W_Q. \end{aligned}$$

This proves (6.4), and thus the lemma. \square

Now we can refine Theorem 5.1 and Corollary 5.5 as

Theorem 6.3. *Let the notation and assumption be as above and let $D = -4nN$. Then $\Xi(n, \mu, v)$ is a Green function for $Z(n, \mu, v)^{\text{Naive}}$ with (at most) log-log singularity, and*

$$dd^c \Xi(n, \mu, v) + \delta_{Z(n, \mu, v)^{\text{Naive}}} = [\omega(n, \mu, v)].$$

Here $\omega(n, \mu, v)$ is the differential defined in (2.8)

$$Z(n, \mu, v)^{\text{Naive}} = \begin{cases} Z(n, \mu) & \text{if } D < 0, \\ g(n, \mu, v) \sum_{M|N} \mathcal{P}_{\frac{1}{M}} & \text{if } D \geq 0 \text{ is a square,} \\ 0 & \text{if otherwise,} \end{cases}$$

and

$$g(n, \mu, v) = \begin{cases} \frac{\sqrt{N}}{4\pi\sqrt{v}} \beta_{3/2}(-4nv\pi) & \text{if } n \neq 0, \mu \notin \frac{1}{2}L/L, \\ \frac{\sqrt{N}}{2\pi\sqrt{v}} \beta_{3/2}(-4nv\pi) & \text{if } n \neq 0, \mu \in \frac{1}{2}L/L, \\ \frac{\sqrt{N}}{2\pi\sqrt{v}} & \text{if } n = 0, \mu = 0, \\ 0 & \text{if } n = 0, \mu \neq 0. \end{cases}$$

Moreover, for every $M|N$,

- (1) when D is not square, the green function $\Xi(n, \mu, v)$ is of exponential decay near cusp $P_{\frac{1}{M}}$.
- (2) When $D = -4Nn > 0$ is a square, one has

$$\Xi(n, \mu, v) = -g(n, \mu, v)(\log |q_M|^2) - 2\psi_M(n, \mu, v; q_M),$$

where q_M is a local parameter at $P_{\frac{1}{M}}$, and $\psi_M(n, \mu, v; q_M)$ is of exponential decay near P_M . Here $P_{\frac{1}{N}} = P_\infty$, and $\psi_N = \psi_\infty$.

(3) When $D = 0$, $\Xi(0, \mu, v) = 0$ when $\mu \notin L$, and

$$\begin{aligned} \Xi(0, 0, v) &= -g(0, 0, v)(\log |q_M|^2) - 2\log(-\log |q_M|^2) \\ &\quad - 2\psi_M(0, \mu, v; q_M), \end{aligned}$$

and

$$\lim_{|q_M| \rightarrow 0} \psi_M(0, 0, v; q_M) = \log \frac{\sqrt{N}}{4\pi\sqrt{v}} - \frac{1}{2}f(0).$$

Here $f(0) = \gamma - \log(4\pi)$ is defined in Lemma 5.2.

6.2. Integral model. Following [KM], let $\mathcal{Y}_0(N)$ ($\mathcal{X}_0(N)$) be the moduli stack over \mathbb{Z} of cyclic isogenies of degree N of elliptic curves (generalized elliptic curves) $\pi : E \rightarrow E'$ such that $\ker \pi$ meets every irreducible component of each geometric fiber. The stack $\mathcal{X}_0(N)$ is regular and proper flat over \mathbb{Z} such that $\mathcal{X}_0(N)(\mathbb{C}) = X_0(N)$. It is a DM-stack. For convenience, we count each point x with multiplicity $\frac{2}{|\text{Aut}(x)|}$ instead of $\frac{1}{|\text{Aut}(x)|}$. It is regular over \mathbb{Z} and smooth over $\mathbb{Z}[\frac{1}{N}]$. When $p|N$, the special fiber $\mathcal{X}_0(N) \pmod{p}$ has two irreducible components \mathcal{X}_p^∞ and \mathcal{X}_p^0 . Both of them are isomorphic to $\mathcal{X}_0(N/p) \pmod{p}$, and they intersect at supersingular points. We require \mathcal{X}_p^∞ to contain the cusp $\mathcal{P}_\infty \pmod{p}$ and \mathcal{X}_p^0 to contain the cusp $\mathcal{P}_0 \pmod{p}$. Here for each divisor $Q|N$, let $\mathcal{P}_{\frac{Q}{N}}$ be the boundary arithmetic curve associated to the cusp $P_{\frac{Q}{N}}$, which is the Zariski closure of $P_{\frac{Q}{N}}$ in $\mathcal{X}_0(N)$ and has a nice moduli interpretation too. We refer to [Co] for detail. It is known that $\mathcal{P}_{\frac{Q}{N}} \pmod{p}$ lies in \mathcal{X}_p^∞ (resp. \mathcal{X}_p^0) if and only if $p \nmid Q$ (resp. $p|Q$).

For $r \in \mathbb{Z}/2N$, $\mu_r = \text{diag}(r/2N, -r/2N) \in L^\sharp/L$ and a positive rational number $n \in Q(\mu_r) + \mathbb{Z}$, let $D = -4Nn \equiv r^2 \pmod{4N}$, $k_D = \mathbb{Q}(\sqrt{D})$ and the order $\mathcal{O}_D = \mathbb{Z}[\frac{D+\sqrt{D}}{2}]$ of discriminant D . When $D < 0$, let $\mathcal{Z}(n, \mu_r)$ be the Zariski closure of $Z(n, \mu_r)$ in $\mathcal{X}_0(N)$. When D is a fundamental discriminant, it has the following moduli interpretation.

Let $\mathfrak{n} = [N, \frac{r+\sqrt{D}}{2}]$, which is an ideal of \mathcal{O}_D with norm N . Following [GZ] and [BY], let $\mathcal{Z}(n, \mu_r)$ be the moduli stack over \mathcal{O}_D of the pairs (x, ι) , where $x = (\pi : E \rightarrow E') \in \mathcal{Y}_0(N)$ and

$$\iota : \mathcal{O}_D \hookrightarrow \text{End}(x) = \{\alpha \in \text{End}(E) : \pi\alpha\pi^{-1} \in \text{End}(E')\}$$

is a CM action of \mathcal{O}_D on x satisfying $\iota(\mathfrak{n}) \ker \pi = 0$. It actually descends to a DM stack over \mathbb{Z} . It is smooth of dimension 1. According to a private note Sanrakan shared with us, the same moduli problem for a general $D < 0$ also produces a flat, horizontal, and regular stack which is the Zariski closure $\mathcal{Z}(n, \mu)$.

The forgetful map

$$\begin{aligned} \mathcal{Z}(n, \mu) &\rightarrow \mathcal{X}_0(N) \\ (\pi : E \rightarrow E', \iota) &\rightarrow (\pi : E \rightarrow E') \end{aligned}$$

is a finite and close map, which is generically 2 to 1. According to [Co, Lemma 2.2 and Remark 2.3], it lies in the regular locus of $\mathcal{X}_0(N)$ and it does not intersect with the boundary of $\mathcal{X}_0(N)$. In particular, one can do arithmetic intersection and height pairing with $\mathcal{Z}(n, \mu_r)$ on $\mathcal{X}_0(N)$ without any problem despite that $\mathcal{X}_0(N)$ is not regular in general ([Co]).

6.3. The metrized Hodge bundle. Let ω_N be the Hodge bundle on $\mathcal{X}_0(N)$ (see [KM]). Then there is a canonical isomorphism $\omega_N^2 \cong \Omega_{\mathcal{X}_0(N)/\mathbb{Z}}(-S)$, which is also canonically isomorphic to the line bundle of modular forms of weight 2 for $\Gamma_0(N)$. Here S is the set of cusps. For a positive integer N , let $\mathcal{M}_k(N)$ be the line bundle of weight k with the normalized Petersson metric

$$\|f(z)\| = |f(z)(4\pi e^{-C}y)^{\frac{k}{2}}|$$

as defined in (1.8). This gives a metrized line bundle $\widehat{\mathcal{M}}_k(N)$ and also induces a metric on ω_N so that the associated metrized line bundle $\widehat{\omega}_N$ satisfies $\widehat{\omega}_N^k \cong \widehat{\mathcal{M}}_k(N)$. From now on, we denote

$$(6.5) \quad k = 12\varphi(N), \quad r = N \prod_{p|N} (1+p^{-1}) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = \frac{3}{\pi} \mathrm{vol}(X_0(N), \mu(z)).$$

Recall that that $\Delta_N(z)$ and $\Delta_N^0(z)$ be both (rational sections) of $\mathcal{M}_k(N)$. The following lemma follows from Propositions 3.3, 3.4, and 3.5, and

$$\Delta_N \mid \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (z) = N^{-6\varphi(N)} \Delta_N^0\left(\frac{z}{N}\right) = N^{-12\varphi(N)} \Pi_{t|N} t^{12a(\frac{N}{t})} \Delta\left(\frac{tz}{N}\right)^{a(\frac{N}{t})}.$$

Lemma 6.4.

$$(6.6) \quad \mathrm{Div} \Delta_N = \frac{rk}{12} \mathcal{P}_\infty - k \sum_{p|N} \frac{p}{p-1} \mathcal{X}_p^0$$

and

$$(6.7) \quad \mathrm{Div} \Delta_N^0 = \frac{rk}{12} \mathcal{P}_0 - \frac{k}{2} \sum_{p|N} \frac{p+1}{p-1} \mathcal{X}_p^\infty - \frac{k}{2} \sum_{p|N} \mathcal{X}_p^0.$$

Here r and k are given by (6.5).

The following lemma is clear.

Lemma 6.5. *Let $q_z = e(z)$ be a local parameter of $X_0(N)$ at the cusp P_∞ .*

- (1) *The metrized line bundle $\widehat{\omega}^k = \widehat{\mathcal{M}}_k(N)$ has log singularity along cusps with all α -index $\alpha_P = \frac{k}{2}$ at every cusp P . At the cusp P_∞ , one has*

$$\|\Delta_N(z)\| = (-\log |q_z|^2)^{\frac{k}{2}} |q_z|^{\frac{r}{12}k} \varphi(q_z),$$

with

$$\varphi(q_z) = e^{-\frac{kC}{2}} \prod_{n=1}^{\infty} |(1 - q_z)^{24C_N(n)}|.$$

- (2) *Both $\widehat{\text{Div}}(\Delta_N) = (\text{Div}(\Delta_N), -\log \|\Delta_N(z)\|^2)$ and $\widehat{\text{Div}}(\Delta_N^0) = (\text{Div}(\Delta_N^0), -\log \|\Delta_N^0(z)\|^2)$ are arithmetic divisors (on $\mathcal{X}_0(N)$) associated to $\widehat{\omega}_N^k$ with log-log singularity at cusps.*

We also consider the arithmetic divisor on $\mathcal{X}_0(N)$:

$$(6.8) \quad \widehat{\Delta}_N = \left(\frac{rk}{12}\mathcal{P}_\infty, -\log \|\Delta_N(z)\|^2\right).$$

One has

$$(6.9) \quad \widehat{\text{Div}}(\Delta_N) = \widehat{\Delta}_N - k \sum_{p|N} \frac{p}{p-1} \mathcal{X}_p^0.$$

Define

$$(6.10) \quad \widehat{\mathcal{Z}}(n, \mu, v) = \begin{cases} \widehat{\mathcal{Z}}(n, \mu, v)^{\text{Naive}} - 2\widehat{\omega}_N - \sum_{p|N} \mathcal{X}_p^0 - (0, \log(\frac{v}{N})) & \text{if } n = 0, \mu = 0, \\ \widehat{\mathcal{Z}}(n, \mu, v)^{\text{Naive}} & \text{otherwise.} \end{cases}$$

The arithmetic generating function ($q = e(\tau)$) in the introduction is defined to be

$$(6.11) \quad \widehat{\phi}(\tau) = \sum_{\substack{n \in \frac{1}{2N}\mathbb{Z} \\ \mu \in L^\sharp/L \\ Q(\mu) \equiv n \pmod{1}}} \widehat{\mathcal{Z}}(n, \mu, v) q_\tau^n e_\mu \in \widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}_0(N), S) \otimes \mathbb{C}[L^\sharp/L][[q]].$$

Replacing $\widehat{\omega}_N$ by the class of arithmetic divisor $\frac{1}{k}\widehat{\text{Div}}(\Delta_N)$, we can rewrite $\widehat{\phi}(\tau) = (\phi(\tau), \Xi_L(\tau, z))$ where

(6.12)

$$\phi(\tau) = \sum_{n, \mu} \mathcal{Z}(n, \mu, v) q^n e_\mu, \quad \text{and}$$

$$\Xi_L(\tau, z) = (\Xi(0, 0, \mu) + \frac{2}{k} \log \|\Delta_N\|^2 - \log \frac{v}{N}) e_0 + \sum_{n \neq 0, \mu} \Xi(n, \mu, v) q^n e_\mu.$$

Notice that

$$(6.13) \quad dd^c \Xi_L(\tau, z) = \Theta_L(\tau, z),$$

which is known by Kudla-Millson a vector valued modular form.

Proposition 6.6. *One has*

$$\widehat{\phi}(\tau) \in \widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}_0(N)) \otimes \mathbb{C}[L^\sharp/L][[q]].$$

Proof. By Theorem 6.3, it suffices to check the case for $\widehat{\mathcal{Z}}(0, 0, v)$. Notice that $\Delta(\tau)$ is a section of ω_N^{12} . So we have by Theorem 6.3

$$\widehat{\mathcal{Z}}(0, 0, v) = (\mathcal{Z}, g)$$

with

$$\begin{aligned} \mathcal{Z} &= -\frac{\sqrt{N}}{2\pi\sqrt{v}} \sum_{M|N} \mathcal{P}_{\frac{1}{M}} - \frac{1}{6} \text{Div } \Delta - \sum_{p|N} \mathcal{X}_p^0, \\ g &= \Xi(0, 0, v) + \frac{1}{6} \log \|\Delta\|^2 - \log \frac{v}{N}. \end{aligned}$$

For each $M|N$, choose $\sigma_M \in \text{SL}_2(\mathbb{Z})$ such that $\sigma_M(\infty) = \frac{1}{M}$. Then Theorem 6.3(3) asserts

$$\begin{aligned} \Xi(0, 0, v)(\sigma_M(z)) &= -\frac{\sqrt{N}}{2\pi\sqrt{v}} (\log |q_M|^2) - 2 \log(-\log |q_M|^2) + \text{smooth} \\ &= -\frac{\sqrt{N}}{2\pi\sqrt{v}} (\log |q|^2) - 2 \log(-\log |q|^2) + \text{smooth}, \end{aligned}$$

where $q_M = q^{\frac{M}{N}}$, as the width of the cusp $P_{\frac{N}{M}}$ is $\frac{N}{M}$. On the other hand, $\log \|\Delta(\sigma_M(z))\|^2 = \log \|\Delta(z)\|^2 = \log(|q|^2) + 12 \log(-\log |q|^2) + \text{smooth}$.

So we know

$$g(\sigma_M(z)) = \left(-\frac{\sqrt{N}}{2\pi\sqrt{v}} + \frac{1}{6}\right) \log(|q|^2) + \text{smooth}$$

has just log singularity. \square

Proposition 6.7. *Let the notation be as above. Then*

$$\deg \widehat{\phi}(\tau) = \langle \widehat{\phi}(\tau), a(2) \rangle = \frac{2}{\varphi(N)} \mathcal{E}_L(\tau, 1).$$

In general, for any $a(f) = (0, f) \in \widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}_0(N), S)$ with S being the set of cusps, $\langle \widehat{\phi}(\tau), a(f) \rangle$ is a vector valued modular form valued in S_L for Γ' of weight $3/2$ and representation ρ_L .

Proof.

$$\langle \widehat{\phi}(\tau), a(f) \rangle = \frac{1}{2} \int_{\mathbb{X}_0(N)} f(z) d_z d_z^c \Xi_L(\tau, z) = \frac{1}{2} \int_{X_0(N)} f(z) \Theta_L(\tau, z)$$

is modular as $\Theta_L(\tau, z)$ is modular. When $f = 2$, one has

$$\deg \widehat{\phi} = \int_{\mathbb{X}_0(N)} \Theta_L(\tau, z) = I(\tau, 1) = \frac{2}{\varphi(N)} \mathcal{E}_L(\tau, 1)$$

by Corollary 2.3. □

Proposition 6.8. *For every prime $p|N$, one has*

$$\langle \widehat{\phi}(\tau), \mathcal{X}_p^0 \rangle = \langle \widehat{\phi}(\tau), \mathcal{X}_p^\infty \rangle = \frac{1}{\varphi(N)} \mathcal{E}_L(\tau, 1) \log p.$$

Proof. Since

$$R(w, w_N z) = R(w_N^{-1} w, z),$$

and $w_N L_\mu = L_{-\mu}$, one has by definition

$$w_N^* \Xi(n, \mu, v) = \Xi(n, -\mu, v) = \Xi(n, \mu, v).$$

This implies

$$w_N^* Z(n, \mu, v)^{\text{Naive}} = Z(n, \mu, v)^{\text{Naive}}$$

on the generic fiber. Since the divisors $Z(n, \mu, v)^{\text{Naive}}$ are all horizontal (flat closure of $Z(n, \mu, v)^{\text{Naive}}$), we have

$$w_N^* \mathcal{Z}(n, \mu, v)^{\text{Naive}} = \mathcal{Z}(n, \mu, v)^{\text{Naive}}.$$

One has also $w_N^* \widehat{\Delta}_N = \widehat{\Delta}_N^0$ and $w_N^* \mathcal{X}_p^0 = \mathcal{X}_p^\infty$. Direct calculation using Lemma 6.4 then shows

$$w_N^* \widehat{\mathcal{Z}}(0, 0, v) = \widehat{\mathcal{Z}}(0, 0, v),$$

and so

$$w_N^* (\widehat{\phi}(\tau)) = \widehat{\phi}(\tau).$$

Since w_N is an isomorphism, we have

$$\begin{aligned} \langle \widehat{\phi}(\tau), \mathcal{X}_p^0 \rangle &= \langle \widehat{\phi}(\tau), \mathcal{X}_p^\infty \rangle \\ &= \frac{1}{2} \langle \widehat{\phi}(\tau), \mathcal{X}_p \rangle = \frac{1}{2} \langle \widehat{\phi}(\tau), (0, \log p^2) \rangle \\ &= \frac{1}{2} \deg \widehat{\phi}(\tau) \log p = \frac{1}{\varphi(N)} \mathcal{E}_L(\tau, 1) \log p. \end{aligned}$$

Here we have used the fact that the principal arithmetic divisor $\widehat{\text{Div}}(p) = (\mathcal{X}_p, -\log p^2)$. This proves the proposition. □

Proof of Theorem 1.3: Now Theorem 1.3 follows from Propositions 6.7 and 6.8, Equation (6.9), and the following theorem, which will be proved in next section.

Theorem 6.9. *Let the notation be above. Then*

$$\langle \widehat{\phi}(\tau), \widehat{\Delta}_N \rangle_{GS} = -12\mathcal{E}'_L(\tau, 1).$$

7. THE PROOF OF THEOREM 6.9

7.1. Some preparation.

Lemma 7.1. *Two different cusps of $X_0(N)$ reduce to two different cusps modulo p for every prime number p . So $\langle \mathcal{P}_{\frac{1}{M_1}}, \mathcal{P}_{\frac{1}{M_2}} \rangle = 0$ if $M_1 \not\equiv M_2 \pmod{N}$.*

Proof. We only need to consider primes $p|N$. If p divides exactly one of the M_1 and M_2 , the two cusps landed in two different branches of \mathcal{X}_p and thus do not coincide. When p divides both of them, their reductions $\bar{\mathcal{P}}_{\frac{1}{M_j}}$ both landed in \mathcal{X}_p^0 . On the other hand, \mathcal{X}_p^0 is isomorphic to the reduction of $\mathcal{X}_0(N/p)$, under which cusps correspond to cusps. Counting the number of cusps, we see that different cusps which landed in \mathcal{X}_p^0 are still different in the reduction. This proves the lemma. \square

Lemma 7.2. *One has for each $p|N$,*

$$\langle \mathcal{X}_p^\infty, \mathcal{X}_p^0 \rangle = -\langle \mathcal{X}_p^0, \mathcal{X}_p^0 \rangle = -\langle \mathcal{X}_p^\infty, \mathcal{X}_p^\infty \rangle = \frac{r(p-1)}{12(p+1)} \log p.$$

Proof. Recall that \mathcal{X}_p^∞ and \mathcal{X}_p^0 are both isomorphic to the special file $\mathcal{X}_0(\frac{N}{p})_p = \mathcal{X}_0(\frac{N}{p}) \pmod{p}$ and that they intersect properly exactly at all the supersingular points. So

$$\begin{aligned} \langle \mathcal{X}_p^\infty, \mathcal{X}_p^0 \rangle &= \sum_{\substack{x \in \mathcal{X}_0(\frac{N}{p})_p(\mathbb{F}_p) \\ \text{supersingular}}} \frac{2}{|Aut(x)|} \\ &= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(\frac{N}{p})] \sum_{\substack{x \in \mathcal{X}_0(1)_p(\mathbb{F}_p) \\ \text{supersingular}}} \frac{2}{|Aut(x)|} \\ &= \frac{r}{p+1} \sum_{\substack{x \in \mathcal{X}_0(1)_p(\mathbb{F}_p) \\ \text{supersingular}}} \frac{2}{|Aut(x)|}. \end{aligned}$$

It is well-known (see for example [KM, corollary 12.4.6])

$$(7.1) \quad \frac{p-1}{24} = \sum_{j \in \overline{F}_p, E_j \text{ supersingular}} \frac{1}{|Aut(E_j)|}.$$

So

$$\langle \mathcal{X}_p^\infty, \mathcal{X}_p^0 \rangle = \frac{r(p-1)}{12(p+1)}.$$

On the other hand

$$\langle \mathcal{X}_p^\infty, \mathcal{X}_p^\infty \rangle + \langle \mathcal{X}_p^\infty, \mathcal{X}_p^0 \rangle = \langle \mathcal{X}_p^\infty, (0, \log p^2) \rangle = 0.$$

So

$$\langle \mathcal{X}_p^\infty, \mathcal{X}_p^\infty \rangle = -\langle \mathcal{X}_p^\infty, \mathcal{X}_p^0 \rangle.$$

□

Lemma 7.3. *One has*

(1)

$$\langle \widehat{\omega}_N, \widehat{\omega}_N \rangle = r\left(\frac{\zeta(-1)}{2} + \zeta'(-1)\right) + \frac{r}{12}C,$$

where $C = \frac{\log(4\pi) + \gamma}{2}$ is the normalization constant in (1.8)

(2)

$$\langle \widehat{\Delta}_N, \widehat{\Delta}_N \rangle = k^2 r \left(\frac{\zeta(-1)}{2} + \zeta'(-1) \right) + \frac{k^2 r C}{12} + \frac{k^2 r}{12} \sum_{p|N} \frac{p^2}{p^2 - 1} \log p.$$

Proof. Let $\widehat{\omega}_{N, \text{Pet}}^k$ be the Hodge bundle with the Petersson metric (via its isomorphism to $\mathcal{M}_k(N)$)

$$\|f(z)\|_{\text{Pet}} = |f(z)(4\pi y)^{\frac{k}{2}}| = \|f(z)\| e^{\frac{kC}{2}}.$$

According to [Kü2, Theorem 6.1], we have

$$(7.2) \quad \langle \widehat{\omega}_{\text{Pet}}, \widehat{\omega}_{\text{Pet}} \rangle = r\left(\frac{\zeta(-1)}{2} + \zeta'(-1)\right).$$

So

$$\begin{aligned} \langle \widehat{\omega}_N, \widehat{\omega}_N \rangle &= \langle \widehat{\omega}_{\text{Pet}}, \widehat{\omega}_{\text{Pet}} \rangle + 2\langle \widehat{\omega}_{\text{Pet}}, (0, C) \rangle \\ &= r\left(\frac{\zeta(-1)}{2} + \zeta'(-1)\right) + \deg(\widehat{\omega}_{\text{Pet}})C \\ &= r\left(\frac{\zeta(-1)}{2} + \zeta'(-1)\right) + \frac{r}{12}C \end{aligned}$$

as claimed.

Next, one has

$$\begin{aligned}
\langle \widehat{\Delta}_N, \widehat{\Delta}_N \rangle &= \langle \widehat{\Delta}_N, \widehat{\text{Div}}(\Delta_N) \rangle + k \sum_{p|N} \frac{p}{p-1} \langle \widehat{\Delta}_N, \mathcal{X}_p^0 \rangle \\
&= \langle \widehat{\text{Div}}(\Delta_N), \widehat{\text{Div}}(\Delta_N) \rangle + k \sum_{p|N} \frac{p}{p-1} \langle \mathcal{X}_p^0, \widehat{\text{Div}}(\Delta_N) \rangle \\
&= k^2 \langle \widehat{\omega}_N, \widehat{\omega}_N \rangle - k^2 \sum_{p|N} \left(\frac{p}{p-1} \right)^2 \langle \mathcal{X}_p^0, \mathcal{X}_p^0 \rangle \\
&= k^2 r \left(\frac{\zeta(-1)}{2} + \zeta'(-1) \right) + \frac{k^2 r C}{12} + \frac{k^2 r}{12} \sum_{p|N} \frac{p^2}{p^2-1} \log p,
\end{aligned}$$

by Lemma 7.2 □

Remark 7.4. We remark that Lemma 7.3 can be proved directly using our explicit description of sections of ω_N^k without using [Kü2, Theorem 6.1]. Indeed, one has

$$\langle \widehat{\omega}_N^k, \widehat{\omega}_N^k \rangle = \langle \widehat{\text{Div}}(\Delta_N), \widehat{\text{Div}}(\Delta_N^0) \rangle.$$

Now direct calculation gives the lemma. We leave the detail to the reader.

7.2. Proof of Theorem 6.9. In this section, we prove Theorem 6.9, which amounts to check term by term on their Fourier coefficients. By Theorem 1.6 and (2.6), it is suffice to prove

$$\langle \widehat{\mathcal{Z}}(n, \mu, v), \widehat{\Delta}_N \rangle = \begin{cases} \int_{X_0(N)} \log \|\Delta_N(z)\| \omega(n, \mu, v) & \text{if } n \neq 0, \\ \int_{X_0(N)} \log \|\Delta_N(z)\| (\omega(0, 0, v) - \frac{dx dy}{2\pi y^2}) & \text{if } n = 0, \mu = 0. \end{cases}$$

The case $n = 0, \mu \neq 0$ is trivial as both sides are zero.

We divide the proof into three cases: D is not a square, $D > 0$ is a square, and $D = 0$.

Case 1: We first assume that D is not a square. In this case, $\mathcal{Z}(n, \mu, v)$ and \mathcal{P}_∞ has no intersection at all. By Proposition 4.1 and Theorem 6.3, one has

$$\langle \widehat{\mathcal{Z}}(n, \mu, v), \widehat{\Delta}_N \rangle = \int_{X_0(N)} \log \|\Delta_N\| \omega(n, \mu, v).$$

This proves the case that D is not square.

Case 2: Now we assume that D is a square. This case is complicated due to self-intersection at \mathcal{P}_∞ . We work out the case $D = 0$ and leave the similar (and slightly easier) case $D > 0$ to the reader. Let

$$\widehat{\mathcal{Z}}_1(0, 0, v) = \widehat{\mathcal{Z}}(0, 0, v)^{\text{Naive}} - \frac{12g(0, 0, v)}{rk} \widehat{\Delta}_N = (\mathcal{Z}_1(0, 0, v), \Xi_1(0, 0, v)).$$

Then

$$\langle \widehat{\mathcal{Z}}(0, 0, v)^{Naive}, \widehat{\Delta}_N \rangle = \langle \widehat{\mathcal{Z}}_1(0, 0, v), \widehat{\Delta}_N \rangle + \frac{12g(0, 0, v)}{rk} \langle \widehat{\Delta}_N, \widehat{\Delta}_N \rangle.$$

We have

$$\begin{aligned} & \langle \widehat{\mathcal{Z}}_1(0, 0, v), \widehat{\Delta}_N \rangle \\ &= \sum_{0 < M|N, M < N} \frac{rk}{12} g(n, \mu, v) \langle \mathcal{P}_{\frac{1}{M}}, \mathcal{P}_\infty \rangle + \frac{rk}{12} (\alpha_{\mathcal{Z}_1, P_\infty} - \psi_{1, \infty}(0, 0, v, 0)) \\ & \quad - \lim_{\epsilon \rightarrow 0} \left(\frac{rk}{12} \alpha_{\mathcal{Z}_1, P_\infty} \log(-\log \epsilon^2) - \frac{1}{2} \int_{X_0(N)_\epsilon} -\log \|\Delta_N\|^2 \omega_1 \right), \end{aligned}$$

where

$$\omega_1 = \omega(0, 0, v) - \frac{12g(0, 0, v)}{r} \frac{dxdy}{4\pi y^2}$$

and

$$\alpha_{\mathcal{Z}_1, P_\infty} = 1 - \frac{6}{r} g(0, 0, v).$$

So the limit is equal to

$$\begin{aligned} & \frac{rk}{12} \alpha_{\mathcal{Z}_1, P_\infty} \lim_{\epsilon \rightarrow 0} \left(\log(-\log \epsilon^2) + \frac{12}{rk} \int_{X_0(N)_\epsilon} \log \|\Delta_N\|^2 \frac{dxdy}{4\pi y^2} \right) \\ & + \lim_{\epsilon \rightarrow 0} \int_{X_0(N)_\epsilon} \log \|\Delta_N\| \left(\omega(0, 0, v) - \frac{dxdy}{2\pi y^2} \right) \\ & = \frac{rk}{12} \alpha_{\mathcal{Z}_1, P_\infty} \lim_{\epsilon \rightarrow 0} \left(\log(-\log \epsilon^2) + \frac{12}{rk} \int_{X_0(N)_\epsilon} \log \|\Delta_N\|^2 \frac{dxdy}{4\pi y^2} \right) \\ & + \int_{X_0(N)} \log \|\Delta_N\| \left(\omega(0, 0, v) - \frac{dxdy}{2\pi y^2} \right). \end{aligned}$$

Recall ([Kü1, Lemma 2.8] that

$$(7.4) \quad \lim_{\epsilon \rightarrow 0} \left(\log(-\log \epsilon^2) + \frac{12}{rk} \int_{X_0(N)_\epsilon} \log \|\Delta_N\|^2 \frac{dxdy}{4\pi y^2} \right) = \frac{r\pi}{3} C_0 + 2 \log(4\pi) - C.$$

Here C_0 is the scattering constant given in Lemma 3.6, and C is the normalization constant in Petersson norm. Combining this with Corollary 3.6, we obtain

$$\begin{aligned} & \langle \widehat{\mathcal{Z}}_1(0, 0, v), \widehat{\Delta}_N \rangle \\ &= \frac{rk}{12} \left(1 - \frac{6}{r} g(0, 0, v)\right) \left(12\zeta(-1) + 24\zeta'(-1) + C + 2 \sum_{p|N} \frac{p^2}{p^2 - 1} \log p\right) \\ & \quad - \frac{rk}{12} \psi_{1,\infty}(0, 0, v, 0) + \int_{X_0(N)} \log \|\Delta_N\| (\omega(0, 0, v) - \frac{dxdy}{2\pi y^2}). \end{aligned}$$

Here we recall $\zeta(-1) = -\frac{1}{12}$. On the other hand, Theorem 5.1 implies

$$\begin{aligned} \psi_{1,\infty}(0, 0, v, 0) &= \lim_{y \rightarrow \infty} (\psi_\infty(0, 0, v, q_z) - \frac{12}{rk} g(0, 0, v) \log \phi(q_z)) \\ &= -\frac{1}{2} \log\left(\frac{v}{N}\right) - \left(1 - \frac{6}{r} g(0, 0, v)\right) C. \end{aligned}$$

Therefore, one has by Lemma 7.3

$$\begin{aligned} & \langle \widehat{\mathcal{Z}}(0, 0, v)^{\text{Naive}}, \widehat{\Delta}_N \rangle \\ &= \langle \widehat{\mathcal{Z}}_1(0, 0, v), \widehat{\Delta}_N \rangle + \frac{12g(0, 0, v)}{rk} \langle \widehat{\Delta}_N, \widehat{\Delta}_N \rangle \\ &= \frac{rk}{24} \log\left(\frac{v}{N}\right) + \frac{2}{k} \langle \widehat{\Delta}_N, \widehat{\Delta}_N \rangle + \int_{X_0(N)} \log \|\Delta_N\| (\omega(0, 0, v) - \frac{dxdy}{2\pi y^2}), \end{aligned}$$

and

$$\begin{aligned} \langle \widehat{\mathcal{Z}}(0, 0, v), \widehat{\Delta}_N \rangle &= \langle \widehat{\mathcal{Z}}(0, 0, v)^{\text{Naive}}, \widehat{\Delta}_N \rangle - \frac{2}{k} \langle \widehat{\Delta}_N, \widehat{\Delta}_N \rangle \\ & \quad - \sum_{p|N} \frac{p+1}{p-1} \langle \mathcal{X}_p^0, \widehat{\Delta}_N \rangle - \langle (0, \log(\frac{v}{N})), \widehat{\Delta}_N \rangle \\ &= \int_{X_0(N)} \log \|\Delta_N\| (\omega(0, 0, v) - \frac{dxdy}{2\pi y^2}). \end{aligned}$$

This proves the case $D = 0$.

8. MODULARITY OF THE ARITHMETIC THETA FUNCTION

In this section, we will prove the modularity of $\widehat{\phi}(\tau)$. To simplify the notation, we denote in this section $X = X_0(N)$ and $\mathcal{X} = \mathcal{X}_0(N)$, and let S be the set of cusps of X . Let g_{GS} be a Gille-Soule Green function for the divisor $\text{Div } \Delta_N$ (without log-log singularity), and let $\widehat{\Delta}_{\text{GS}} = (\text{Div } \Delta_N, g_{\text{GS}}) \in \widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X})$, and $f_N = g_{\text{GS}} + \log \|\Delta_N\|^2$. Then

$$a(f_N) = (0, f_N) \in \widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}, S) \text{ and}$$

$$\widehat{\Delta}_{\text{GS}} = \widehat{\text{Div}}(\Delta_N) + a(f_N).$$

Theorem 1.3 implies the following proposition immediately.

Proposition 8.1. *The Gille-Soule height pairing $\langle \widehat{\phi}, \widehat{\Delta}_{\text{GS}} \rangle$ is a vector valued modular form of Γ' valued in $\mathbb{C}[L^\sharp/L]$ of weight $3/2$ and representation ρ_L .*

Now the proof of Theorem 1.1 follows the same way as in [KRY2, Chapter 4] with $\widehat{\omega}$ there replaced by $\widehat{\Delta}_{\text{GS}}$. For the convenience of the reader and for completeness, we sketch the proof here and refer to *loc. cit.* for detail. Let $\mu_{\text{GS}} = c_1(\widehat{\Delta}_{\text{GS}})$. Let $A(X)$ be the space of smooth functions f on X which are conjugation invariant (Frob_∞ -invariant), and let $A^0(X)$ be the subspace of functions $f \in A(X)$ with

$$\int_X f \mu_{\text{GS}} = 0.$$

For each $p|N$, let $\mathcal{Y}_p = \mathcal{X}_p^\infty - p\mathcal{X}_p^0$, then $\langle \mathcal{Y}_p, \widehat{\Delta}_{\text{GS}} \rangle = 0$. Let $\mathcal{Y}_p^\vee = \frac{1}{\langle \mathcal{Y}_p, \mathcal{Y}_p \rangle} \mathcal{Y}_p$. Finally let $\widetilde{\text{MW}}$ be the orthogonal complement of $\mathbb{R}\widehat{\Delta}_{\text{GS}} + \sum_{p|N} \mathbb{R}\mathcal{Y}_p^\vee + \mathbb{R}a(1) + a(A^0(X))$ in $\widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X})$. Then one has

Proposition 8.2. ([KRY2, Propositions 4.1.2, 4.1.4] *One has*

$$\widehat{\text{CH}}_{\mathbb{R}}^1(\mathcal{X}) = \widetilde{\text{MW}} \oplus (\mathbb{R}\widehat{\Delta}_{\text{GS}} + \sum_{p|N} \mathbb{R}\mathcal{Y}_p^\vee + \mathbb{R}a(1)) \oplus a(A^0(X)).$$

More precisely, every $\widehat{Z} = (\mathcal{Z}, g_Z)$ decomposes into

$$\widehat{Z} = \widetilde{Z}_{\text{MW}} + \frac{\deg \widehat{Z}}{\deg \widehat{\Delta}_{\text{GS}}} \widehat{\Delta}_{\text{GS}} + \sum_{p|N} \langle \widehat{Z}, \mathcal{Y}_p \rangle \mathcal{Y}_p^\vee + 2\kappa(\widehat{Z})a(1) + a(f_{\widehat{Z}})$$

for some $f_{\widehat{Z}} \in A^0(X)$, where

$$\kappa(\widehat{Z}) \deg \widehat{\Delta}_{\text{GS}} = \langle \widehat{Z}, \widehat{\Delta}_{\text{GS}} \rangle - \frac{\deg \widehat{Z}}{\deg \widehat{\Delta}_{\text{GS}}} \langle \widehat{\Delta}_{\text{GS}}, \widehat{\Delta}_{\text{GS}} \rangle.$$

Proposition 8.3. ([KRY2, Remark 4.1.3]) *Let $\text{MW} = J_0(N) \otimes_{\mathbb{Z}} \mathbb{R}$. Then $\widehat{Z} = (\mathcal{Z}, g_Z) \mapsto Z - \frac{\deg \widehat{Z}}{\deg \widehat{\Delta}_{\text{GS}}} \text{Div}(\Delta_N)_{\mathbb{Q}} \in \text{MW}$ induces an isomorphism*

$$\widetilde{\text{MW}} \cong \text{MW}.$$

Here $\text{Div}(\Delta_N)_{\mathbb{Q}}$ is generic fiber of $\text{Div}(\Delta_N)$, and Z is the generic fiber of \mathcal{Z} . The inverse map is given as follows. Given a rational divisor

$Z \in J_0(N)$, let g_Z be the unique harmonic Green function for Z such that

$$d_z d_z^c g_Z - \delta_Z = 0, \\ \int_X g_Z \mu(Z) = 0.$$

Let \mathcal{Z} be a divisor of \mathcal{X} with rational coefficients such that its generic fiber is Z , and it is orthogonal to every irreducible components of the closed fiber \mathcal{X}_p for each prime p . Finally let

$$\tilde{\mathcal{Z}} = \widehat{\mathcal{Z}} - 2a(\langle \widehat{\Delta}_{\text{GS}}, \widehat{\mathcal{Z}} \rangle), \quad \widehat{\mathcal{Z}} = (\mathcal{Z}, g_Z).$$

Then the map $Z \mapsto \tilde{\mathcal{Z}}$ is the inverse isomorphism.

Finally, let Δ_z be the Laplacian operator with respect to μ_{GS} . Then the space $A^0(X)$ has an orthonormal basis $\{f_j\}$ with

$$\Delta_z f_j + \lambda_j f_j = 0, \quad \langle f_i, f_j \rangle = \delta_{ij}, \quad \text{and } 0 < \lambda_1 < \lambda_2 < \cdots,$$

where the inner product is given by

$$\langle f, g \rangle = \int_{X_0(N)} f \bar{g} \mu_{\text{GS}}.$$

In particular, every $f \in A^0(X)$ has the decomposition

$$(8.1) \quad f(z) = \sum \langle f, f_j \rangle f_j.$$

Recall also ([KRY2, (4.1.36)]) that

$$(8.2) \quad d_z d_z^c f = \Delta_z(f) \mu_{\text{GS}}.$$

With the above preparation, we are now ready to restate Theorem 1.1 in a slightly more precise form as follows.

Theorem 8.4. *Let the notation be as above. Then*

$$\widehat{\phi}(\tau) = \tilde{\phi}_{\text{MW}}(\tau) + \phi_{\text{GS}}(\tau) \widehat{\Delta}_{\text{GS}} + \sum_{p|N} \phi_p(\tau) \mathcal{Y}_p^\vee + \phi_1 a(1) + a(\phi_{\text{SM}})$$

where ϕ_p , ϕ_1 , and ϕ_{GS} are real analytic modular forms of Γ' of weight $3/2$ and representation ρ_L valued in $\mathbb{C}[L^\sharp/L]$, $\tilde{\phi}_{\text{MW}}(\tau)$ is a modular form of Γ' of weight $3/2$ and representation ρ_L valued in finite dimension vector space $\widehat{MW} \otimes \mathbb{C}[L^\sharp/L]$, and ϕ_{SM} is a modular form of Γ' of weight $3/2$ and representation ρ_L valued in $A^0(X_0(N)) \otimes \mathbb{C}[L^\sharp/L]$.

Proof. Under the isomorphism in Propositions 8.3, $\tilde{\phi}_{MW}$ becomes (here we use Manin's well-known result that the divisor of degree 0 supported on cusps is torsion and is thus zero in $\mathrm{CH}_{\mathbb{R}}^1(X)$)

$$\phi(\tau)_{\mathbb{Q}} - \frac{\deg \hat{\phi}}{\deg \hat{\Delta}_{\mathrm{GS}}} \mathrm{Div}(\Delta_N)_{\mathbb{Q}} = \sum_{n>0, \mu} (Z(n, \mu) - \deg Z(n, \mu) P_{\infty}) q^n e_{\mu}$$

is modular by either the main result of Gross-Kohnen-Zagier [GKZ] (note that Jacobi forms there are the same as vector valued modular forms we used here), or Borcherds' modularity result for $\phi(\tau)_{\mathbb{Q}}$ (see [KRY2, Theorem 4.5.1]) and Proposition 6.7. The modularity of ϕ_p , ϕ_1 and ϕ_{GS} is given by Propositions 8.2, 6.8, 6.7, and 8.1.

Finally, one has by (8.1),

$$\phi_{SM}(\tau, z) = \sum_j \langle \phi_{SM}, f_j \rangle f_j.$$

Simple calculation gives

$$\begin{aligned} \langle \phi_{SM}, f_j \rangle &= -\frac{1}{\lambda_j} \int_{X_0(N)} \phi_{SM}(\tau, z) \Delta_z(\bar{f}_j) \mu_{\mathrm{GS}} \\ &= -\frac{1}{\lambda_j} \int_{X_0(N)} \phi_{SM}(\tau, z) d_z d_z^c \bar{f}_j \\ &= -\frac{1}{\lambda_j} \int_{X_0(N)} d_z d_z^c \phi_{SM}(\tau, z) \bar{f}_j. \end{aligned}$$

One has by Proposition 8.2,

$$\phi_{SM} = \Xi(\tau, z) - g_{MW} - \phi_{\mathrm{GS}}(\tau) g_{\mathrm{GS}} - \phi_1(\tau),$$

where g_{MW} is the harmonic Green function in $\tilde{\phi}_{MW}$. So

$$\begin{aligned} -\lambda_j \langle \phi_{SM}, f_j \rangle &= \int_{X_0(N)} d_z d_z^c \Xi(\tau, z) f_j - \phi_{\mathrm{GS}}(\tau) \int_{X_0(N)} d_z d_z^c g_{\mathrm{GS}} f_j \\ &= \langle \hat{\phi}(\tau), a(f_j) \rangle - \phi_{\mathrm{GS}}(\tau) \int_{X_0(N)} d_z d_z^c g_{\mathrm{GS}} f_j \end{aligned}$$

is modular by Proposition 6.7. Therefore ϕ_{SM} is modular. \square

This proof was explained to one of us by Sid Sankaran and avoids the metrized line bundle with log singularity and Eisenstein series in the usual spectral decomposition of $A^0(X)$ with respect to $\mu(z) = \frac{dx dy}{y^2} = 4\pi c_1(\hat{\omega}_N)$. We thank him for kindly sharing his idea and other help.

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